Mathematical Reviews

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HISTORY

Irani, Rida A. K. A sexagesimal multiplication table in the Arabic alphabetical system. Scripta Math. 18, 92–93 (1952).

Rajagopal, C. T., and Vedamurti Aiyar, T. V. A Hindu approximation to pi. Scripta Math. 18, 25–30 (1952).

Zubov, V. P. On the character of the old Russian mathematics. Uspehi Matem. Nauk (N.S.) 7, no. 3(49), 83–96 (1952). (Russian)

The author undertakes to show, in contrast to the customary notion that old Russian mathematics was entirely of a primitive character stemming from practical problems, that it was in fact concerned also with ancient and medieval discussions of the continuum and related problems. He adduces selections from a number of manuscripts, frequently giving the Old Russian text and a translation into modern Russian in parallel columns, to demonstrate that Aristotelian concepts penetrated into medieval Russia by way of the Dialectics of John of Damascus (fl. c. 730) and other oriental authors, and also through Western European sources. *E. S. Kennedy* (Beirut).

Poletti, Luigi. Il contributo italiano alla tavola dei numeri primi. Tavola dell'undicesimo milione. Rivista Mat. Univ. Parma 2, 417–434 (1951).

This paper gives a history of lists of primes with particular emphasis upon the lists of primes of the form ax^2+bx+c which the author has been compiling for many years. There is also an account of the recent list of primes of the eleventh million of Kulik, Poletti and Porter [Amsterdam, 1951; these Rev. 13, 625]. *D. H. Lehmer.*

Kraitchik, Maurice. On the factorization of $2^n\pm1$. Scripta Math. 18, 39–52 (1952).

The author gives a history of the methods of factoring the numbers of Fermat and Mersenne and the more general numbers $2^n\pm1$ from the time of Euclid to 1951. A table of the complete factorizations of $2^n\pm1$ as far as known in 1951 is given together with a small table of the coefficients of the polynomials X_n and Y_n in the identity of Aurifeuille: $a^n+b^n=X_n^2+nabY_n^2$. *D. H. Lehmer.*

Szőkefalvi-Nagy, Béla. New results in analysis. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 2 (1951), 59–71 (1952). (Hungarian)

The author reviews the results in analysis obtained by Hungarian mathematicians in 1951. *P. Erdős.*

Szele, Tibor. New results in abstract algebra. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 2 (1951), 73–87 (1952). (Hungarian)

An expository lecture describing some of the recent work of Hungarian algebraists. *P. R. Halmos.*

Hajós, György. New results in geometry. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 2 (1951), 119–123 (1952). (Hungarian)

The author reviews the results in geometry obtained by Hungarian mathematicians in the year 1951. *P. Erdős.*

Rényi, Alfréd. New results in probability theory. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 2 (1951), 125–139; discussion 140–144 (1952). (Hungarian)

This is a survey of the work performed in Hungary during the last year. *E. Lukacs* (Washington, D. C.).

Kuratowski, K. Report on the scientific activities of the Polish State Mathematical Institute, especially in the field of topology. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 2 (1951), 113–118 (1952). (Hungarian)

Bar-Hillel, Y. Bolzano's propositional logic. Arch. Math. Logik Grundlagenforsch. 1, 65–98 (1952).

Sakellariou, Nilos. Obituary: Constantin Carathéodory. Bull. Soc. Math. Grèce 26, 1–13 (1952). (Greek)

Campedelli, Luigi. Obituary: Guido Castelnuovo. Boll. Un. Mat. Ital. (3) 7, 241–246 (1952).

★Klero, A. Teoriya figury Zemli, osnovannaya na načalah gidrostatiki. [Clairaut, A. Théorie de la figure de la Terre tirée des principes de l'hydrostatique.] Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1947. 358 pp. (4 plates). 23 rubles.

In addition to the Russian translation of Clairaut's work [first edition, Paris, 1743], the volume contains an extract from D'Alembert's "Sur l'équilibre des fluides" [Opus. Math., v. 5, pp. 10–22, Briasson, Paris, 1768], two letters of Voltaire, an essay by N. I. Idel'son on this work of Clairaut's, and a commentary on the work. The translation is by N. S. Yahontova.

Godeaux, Lucien. Obituary: Notice sur Alphonse Demoulin. Ann. Acad. Roy. Belgique. Notices Biographiques 118, 3–35 (1952).

★Kovalevskaya, S. V. Naučnye raboty. [Scientific works.] Editing and commentary by P. Ya. Polubarinova-Kočina. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1948. 368 pp. (2 plates). 22.50 rubles.

This volume contains all of Kovalevskaya's scientific papers in Russian translation. In appendices several of the papers are discussed with the aim of clarifying them and relating them to contemporary and later work. In addition, there are a biography outlining particularly her scientific activity, translations of several letters of Weierstrass concerning her, a list of her literary works, and a bibliography of books concerning her.

Polubarinova-Kočina, P. Ya. From the correspondence of S. V. Kovalevskaya. Uspehi Matem. Nauk (N.S.) 7, no. 4(50), 103–125 (1952). (Russian)

A selection of Kovalevskaya's correspondence (translated into Russian) with Mittag-Leffler, Čebyšev, Hermite, Cantor, Kronecker and others.

Sedov, L. I. Basic data of the life and activity of L. S. Leibenzon. Uspehi Matem. Nauk (N.S.) 7, no. 4(50), 127–134 (1 plate) (1952). (Russian)

Bari, N. K., and Lyusternik, L. A. The work of N. N. Luzin on the metric theory of functions. Uspehi Matem. Nauk (N.S.) 6, no. 6(46), 28–46 (1951). (Russian)

A bibliography of both Luzin's work and related work of other authors is included.

Truesdell, C. A. Obituary: Paul Felix Neményi: 1895–1952. Science (N.S.) 116, 215–216 (1952).

Taussky-Todd, Olga. Arnold Scholz zum Gedächtnis. Math. Nachr. 7, 379–386 (1952).

Arnold Scholz was born 24 December 1904 and died 1 February 1942. In addition to a discussion of his work a list of his published papers is given.

★N'yuton, Isaak. Vseobščaya arifmetika ili kniga ob arifmetičeskih sinteze i analize. [Newton, Isaac. Arithmetica universalis, sive de compositione et resolutione arithmetica liber.] Translation, essay and commentary by A. P. Yuškevič. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1948. 442 pp. 25 rubles.

The Russian translation was made from the 1732 edition of the "Arithmetica universalis" [Leiden, 1732] and collated with Raphson's English translation "Universal arithmetick: or, a treatise of arithmetical composition and resolution" [London, 1728].

★Uryson, P. S. Trudy po topologii i drugim oblastyam matematiki. [Works on topology and other fields of mathematics.] Gosudarstv. Izdat. Tehn.-Teoret. Lit., Moscow-Leningrad, 1951. Vol. I, pp. 1–512; vol. II, pp. 513–992. 19.75+18.65 rubles.

These two volumes include all of Uryson's mathematical papers except short notes announcing results published later in longer papers. There is also an essay by P. S. Aleksandrov, "P. S. Uryson and his place in mathematical science". Short notes, mostly by Aleksandrov, follow the individual papers, all of which are translated into Russian if originally in another language.

FOUNDATIONS

Behmann, Heinrich. Das Auflösungsproblem in der Klassenlogik. Arch. Math. Logik Grundlagenforsch. 1, 17–29 33–51 (1950).

The "Auflösungsproblem", which forms the subject of this paper, may be described as follows: let

$$F(A, B, C, \cdots; X, Y, \cdots; a, b, c, \cdots)$$

denote a propositional function formed by purely logical means from the free one-place predicate variables $A, B, C, \cdots, X, Y, \cdots$ and, perhaps, the free individual variables $a, b, \cdots$; it is required to exhibit, in some explicit and systematic matter, the values of the $X, Y, \cdots$ (as functions of the $A, B, C, \cdots, a, b, c, \cdots$) which make F true. As "purely logical means" the author admits: (1) the application of a predicate to an argument to form a propositional form; (2) the formation of propositional forms from the equality relation and individual variables; (3) the combination of propositional forms by the connectives of classical propositional algebra; (4) quantification with respect to individual variables; and (5) quantification with respect to one-place predicate variables. The author shows this problem is theoretically capable of finite solution. The basic principle of the method is that F can be regarded as a disjunction of assignments of numerical quantifications to the elementary cells into which a development (in Schröder's sense) relative to the predicate-variables divides the universe. The paper is in a sense a continuation of the author's paper on the decision problem [Math. Ann. 86, 163–229 (1922)]. (In the introduction the author points out a connection with the arithmetical methods of Boole as opposed to those of Jevons and Schröder; this suggests that methods similar to those of Hoff-Hansen [Norsk Mat. Tidsskr. 25, 6–12 (1943); these Rev. 8, 125] and Skolem [ibid. 25, 13–16 (1943); these Rev. 8, 125] might possibly be of use in special cases.)

H. B. Curry (State College, Pa.).

Behmann, Heinrich. Zu den Parallelreihentransformationen in Schröders "Algebra und Logik der Relative." Arch. Math. Logik Grundlagenforsch. 1, 52–62 (1951).

This relates to a special problem of the algebra of relations which was discussed by E. Schröder in the third volume of his Vorlesungen über die Algebra der Logik [Teubner, Leipzig, 1895], pp. 140–144, 201–240. The author shows that the problem reduces to a problem of the algebra of classes, and relates it to his solution of the "Auflösungsproblem" (see the preceding review). *H. B. Curry.*

Curry, Haskell B. On the definition of negation by a fixed proposition in inferential calculus. J. Symbolic Logic 17, 98–104 (1952).

If LX is one of the L-systems of logic introduced in the author's book "A theory of formal deducibility" [Notre Dame, 1950; these Rev. 11, 487], a system LXF is obtained by dropping negation as a primitive concept, adjoining a primitive proposition F, defining negation by $\neg A =_{D} A \supset F$, and suitably modifying the rules. In an obvious way, to any elementary statement Γ in LX, an elementary statement Γ_F in LXF is constructed, such that Γ is valid in LX if and only if Γ_F is valid in LXF. The proof of the "if" encountered some difficulties which the author overcomes here.

A. Heyting (Amsterdam).

Kreisel, G. On the interpretation of non-finitist proofs. I. J. Symbolic Logic 16, 241–267 (1951).

The author discusses free-variable interpretations of various extensions of the first-order predicate calculus with the ϵ-formula and axioms for equality [D. Hilbert and P. Bernays, Grundlagen der Mathematik, vol. 2, Springer, Berlin, 1939]. An interpretation of a system is a recursive function $f(n, a)$ such that: (1) For every formula $\mathfrak{A}$ of Σ, $f(n, a)$ is the Gödel number of a free-variable formula A_n. (2) From a proof of $\mathfrak{A}$ in Σ, there can be found an $\mathfrak{n}$ such that $A_{\mathfrak{n}}$ is verifiable. (3) If $\neg\mathfrak{A}$ is provable in Σ, then for each n,

there is a substitution for individual and function variables of A_n such that A_n is false. (4) If $\mathfrak{B}$ is deducible from $\mathfrak{A}$ in Σ, then there exists a recursive $g(n)$ such that $B_{g(n)}$ is verifiable if A_n is verifiable. In the simplest of the interpretations, A_n represents "m is not the Gödel number of a proof of $\neg\mathfrak{A}$ in Σ", with m as a free variable. In another interpretation, the A_n are disjunctions specified by Herbrand's Theorem [see Hilbert-Bernays, op. cit., p. 158]. Finally, an interpretation is afforded by the first ϵ-theorem of Hilbert-Bernays. This interpretation shows that for any set of functions and constants which might be proposed to satisfy $\neg\mathfrak{A}$, a counter-example may be furnished in terms of functionals of the predicate calculus if $\mathfrak{A}$ is provable. The author then outlines the extension of this "no-counter-example" interpretation to the system of number theory Z of Hilbert-Bernays and extensions of it, reserving for the second part of the paper the details of the demonstration.

D. Nelson (Washington, D. C.).

Wang, Hao. Truth definitions and consistency proofs. Trans. Amer. Math. Soc. **73**, 243–275 (1952).

Notation. Ordinary, primed, starred letters denote expressions of systems (S), (S'), (A). Here (A) is a system of recursive number theory containing the symbols $0, 0+1, \cdots$ ('numerals'). (S) is a set theory with a 'model' for (A), i.e., theorems of (A) go into theorems of (S) if their non-logical constants are replaced by suitable expressions of (S) and the variables are required to satisfy N; 0 of (A) is replaced by s_0, x_*+1 by $s(x)$, and, generally, M_* by M; s_{n+1} denotes $s(s_n)$, $\vdash M$ means: M can be proved in (S). An arithmetization of the syntax of (S') is chosen, the numeral M_*' is the Gödel-number of M'; for numerals $m_*, \cdots, q_*$,

$$\vdash[C_*(m_*, n_*, p_*)\&E_*(m_*, q_*)]$$

if and only if: $\vdash_A m_* = M_*', \cdots, \vdash' P'$ derived in one step from $\vdash' M'$ and $\vdash' N'$, Q' denotes $-M'$. $\mathrm{Con}_*(S)$ [$\mathrm{Con}_*(S')$] is arithmetization in (A) of: (S) [(S')] is consistent. The expressions C_*, E_*, $\mathrm{Con}_*(S)$, $\mathrm{Con}_*(S')$ are naturally not uniquely determined by these conditions, the author's choice makes Gödel's second undecidability theorem applicable.

Facts. (a) There are terms i for which $\vdash N(i)$, but not: $\vdash s_0 = i$, or $\vdash s_1 = i$, or $\cdots$ (Gödel). (b) If $I(\phi)$ means

$$\phi(s_0)\&.\bigwedge_{N(x)} \phi(x)\rightarrow\phi[s(x)]:\rightarrow \bigwedge_{N(x)} \phi(x),$$

$\vdash I(\phi)$ need hold only for predicates ϕ_* of (A); if $\vdash I(\phi)$ for each ϕ of (S) without bound class variables, we have 'set induction'. (c) Three precise forms of naïve conditions on truth-definition Tr for (S') in (S), i.e., 'M' equivalent to $\mathrm{Tr}\,(s_{M_*'})$': (α) For suitable (S'') embracing both (S) and (S'), and any M', $\vdash''[M' \leftrightarrow \mathrm{Tr}\,(s_{M_*'})]$; ($\beta$) for any given numerals $m_*, \cdots, q_*$, (i) $\mathrm{Tr}\,(s_{P_*'})$, P' axiom of (S'),

$$\text{(ii)}\quad \vdash[C(s_{m_*}, s_{n_*}, s_{p_*})\&E(s_{p_*}, s_{q_*})\&\mathrm{Tr}\,(s_{m_*})\&\mathrm{Tr}\,(s_{n_*}). \rightarrow. \mathrm{Tr}\,(s_{p_*})\&-\mathrm{Tr}\,(s_{q_*})];$$

(γ) as (β), but instead of $s_{m_*}, \cdots, s_{q_*}$ arbitrary terms $i, \cdots$, with $\vdash N(i)$ (cf. (a)), and (iii) $I(\mathrm{Tr})$. By least number principle γ(ii) is equivalent to quantification over $N(x)$. Then (γ) yields $\vdash \mathrm{Con}(S')$.

Results. A fairly detailed truth-definition (γ) for (S_2), of Zermelo type, in (S_1), a system similar to that of Quine [J. Symbolic Logic **6**, 135–149 (1941); these Rev. **3**, 289]. Further, truth-definitions (α) and (β) for typical systems (S), (S') where the consistency of (S) follows from that of (S') by elementary means, actually $\vdash[\mathrm{Con}(S')\rightarrow\mathrm{Con}(S)]$. Two of the conditions γ(i)-γ(iii) are checked, so (Gödel), if the third were satisfied, (S) would be inconsistent: a new batch of undecidable formulae, but cf. Mostowski [Fund. Math. **37**, 111–124 (1950); these Rev. **12**, 791]. Special cases are stressed: if (S') has only a finite number of axioms, γ(i) follows from β(i), if Tr does not contain bound class variables and set induction applies to (S), then γ(iii) holds. For (α) and (β): (S) need not be 'stronger' than (S'), but must have additional symbols. A truth-definition (α) in (S) is a truth-definition for any extension $(S^\dagger)$ of (S') obtained by adding new (consistent) axioms, but no new symbols; thus (S) may be 'weaker' than $(S^\dagger)$.

Remarks on truth-definitions for many-sorted theories, and a 'question': can one 'express' the consistency of (S) by $\mathrm{Con}_1(S)$ such that $\vdash \mathrm{Con}_1(S)$? Positive answer: Gentzen [Math. Ann. **112**, 493–565 (1936)] associates ordinals $<\epsilon$ with proofs of a number theory (Z) such that: $\vdash_Z$ 'proofs with finite ordinals are consistent', and $\vdash_Z$ 'for infinite α, if a numerical formula can be proved by an α-proof, then also by a β-proof with $\beta<\alpha$'. If transfinite induction up to ϵ is accepted, this 'expresses' the consistency of (Z).

G. Kreisel (Reading).

Rose, Alan. An extension of the calculus of non-contradiction. Proc. London Math. Soc. (2) **54**, 184–200 (1952).

Verf. überträgt die Methode von G. E. Dexter [Amer. J. Math. **65**, 171–178 (1943); diese Rev. **4**, 126], den 2-wertigen Aussagenkalkül mit einer primitiven Funktion $n(p, q, \cdots)$, d.h., non(p et q et $\cdots$), ohne Axiome auf einer einzigen Regel aufzubauen, auf die 3-wertige Logik. Die Definition der Funktionen von Łukasiewicz-Słupecki lauten

$$Np = n^2n^2np.n^2p..npnp...$$
$$Cpq = Nn^3pNq.Nn^2n^2p.n^2q.....$$
$$Tp = n^2pnpn^2p..$$

wenn $np.$ bzw. $npq.$ statt $n(p)$ bzw. $n(p, q)$ geschrieben wird und $n^2p.$ bzw. $n^3p.$ statt $nnp..$ bzw. $nnnp...$.

P. Lorensen (Bonn).

Fletcher, T. J. The solution of inferential problems by Boole algebra. Math. Gaz. **36**, 183–188 (1952).

★Goodstein, R. L. Constructive formalism. Essays on the foundations of mathematics. University College, Leicester, 1951. 91 pp. 15s.

This collection of essays outlines the author's views of the importance of a finitary mathematics and presents an approach more narrowly constructivist than that of the intuitionists. In the introduction he asserts "the constructivists deny and the formalists affirm the possibility of completing an endless process". While this is perhaps a harsh judgement of the formalist, it does state the central divergence between constructive and classical mathematics. Of the number concept, the author says, "The concepts of number and function are defined by the transformation rules for number and function signs. It is not a one-to-one related pair of classes that determines a function, but a function which determines a one-to-one pair of related classes."

A finitary system of arithmetic was first proposed by Skolem [Skr. Vid. Kristiania. I. Mat.-Nat. Kl. **1923**, no. 6] and the author's equation calculus encompasses essentially the same portion of arithmetic. This equation calculus [Proc. London Math. Soc. (2) **48**, 401–434 (1945); these Rev. **8**, 245] and its application to the foundation of various

parts of mathematics is the central concern of the work. Of it the author says: ". . . we have attempted a resolution of the formalist-finitist controversy in the foundations of mathematics by the construction of an axiom-free equation calculus which provides a purely formal criterion for the intuitive concept of a finitist proof. But the reduction of mathematics to a sign language does not, of itself, contribute towards a solution of the deepest problems about the nature of signs and the relation of language to reality." In a final chapter discussing some of these basic problems, the author suggests that a language is related to experience in much the way that one language is related to another by translation, experience being a kind of "real-object" language.

A chapter on the consistency problem contains a brief treatment of Gödel's work on the incompleteness of arithmetic. It should be noted that the demonstration outlined depends on the concept of omega-consistency although this notion is not explicitly introduced. In this critique the author attacks the use of reductio ad absurdum as a metamathematical method: among other remarks, "We must either abandon the method of derivation from a contradiction or the formalization of the number concept." Further chapters discuss the theory of constructive ordinals and an approach to constructive concepts of length and area. While many of the author's statements are not likely to meet universal agreement, the book is a valuable and stimulating contribution. *D. Nelson* (Washington, D. C.).

Hasenjaeger, G. Über ω-Unvollständigkeit in der Peano-Arithmetik. J. Symbolic Logic 17, 81–97 (1952).

Let P be the first order predicate calculus extended with Peano's axioms. From P any formula $(x)(x\equiv\mathbf{n}\rightarrow x'\equiv\mathbf{n}')$, where $\mathbf{n}$ is a numeral, is derivable; however,

$$H\equiv(y)(x)(x=y\rightarrow x'=y')$$

is not derivable. The latter result is proved by the theory of models, for which the author gives an abstract formulation. In his terminology, a normal semimodel $\mathfrak{M}$ for P is such that the axioms and rules of P are valid in $\mathfrak{M}$ with the possible exception of the substitution rule for predicates. If the latter is also valid, $\mathfrak{M}$ is called P-closed; if the set of admissable predicates comprises all predicates, $\mathfrak{M}$ is called absolute. P is categorical in the following sense: any two absolute semimodels of P are isomorphic. On the other hand, a closed semimodel is given, in which H is not valid; this proves that H is not derivable in P. H becomes derivable if the second order predicate calculus is used. Analogous results are derived for a system of axioms indicated by Lorenzen: $a\equiv a$; $a\equiv b\rightarrow(Fa\rightarrow Fb)$; $a\neq 0\rightarrow(Ex)(x'=a)$; $Fa\rightarrow(Ex)(Fx\wedge(y)(Fy\rightarrow y'\neq x))$. From this system the Peano axioms are derivable with the exception of $a'\equiv b'\rightarrow a\equiv b$; more exactly, $(x)(x'\equiv\mathbf{n}'\rightarrow x\equiv\mathbf{n})$ is derivable for any numeral $\mathbf{n}$, but $(y)(x)(x'\equiv y'\rightarrow x\equiv y)$ is not derivable.
A. Heyting (Amsterdam).

Lombardo-Radice, Lucio. Ordinali transfiniti e principio del terzo escluso (a proposito di un ragionamento del Gödel). Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 9, 421–428 (1950).

The author confuses Brouwer's rejection of the principle of excluded middle with Gödel's theorem about undecidable propositions in formal systems. He argues against the proof of the latter, as exposed by Gödel in his introduction [Monatsh. Math. Phys. 38, 173–198 (1931), p. 175], that the class-defining formula S cannot belong to the sequence $R(n)$ because its definition presupposes this sequence as a whole; in reality, S is defined by formal operations from the formula defining the order relation R. *A. Heyting*.

✱Greenwood, Thomas. La nature du transfini. Les Éditions de l'Université d'Ottawa, Ottawa, 1946. 68 pp.

This monograph is divided into six chapters, of which the first and last embody historical and philosophical commentary; the intervening chapters are devoted mainly to the technical material. Chapter II discusses abstract sets and operations therewith; Chapter III, cardinal numbers and operations; Chapter IV, ordinal numbers; and Chapter V comprises a brief description of some classical applications in transfinite arithmetic, integration, and topology. This material is probably intended only as a basis for the philosophical discussion, and not as a text, since it omits some of the definitions and methods indispensable for the mathematics student. There are also some errors and omissions in basic notions. For instance, one of the Peano postulates for the natural numbers is missing in the listing on pp. 24–25; on p. 46 it is asserted that the choice axiom allows of proving that every transfinite cardinal is an aleph, "which confirms the continuum hypothesis;" and on p. 52 the definition sketched of the Lebesgue integral seems misleading.

The "philosophical conclusions" of the last chapter, affirming the legitimacy of the transfinite numbers, emphasize the thomistic point of view. Brief paragraphs are devoted to the ideas of Hilbert and Brouwer.
R. L. Wilder (Ann Arbor, Mich.).

ALGEBRA

Andree, Richard V. Cryptanalysis. Scripta Math. 18, 5–15 (1 plate) (1952).

Bose, R. C., and Connor, W. S. Combinatorial properties of group divisible incomplete block designs. Ann. Math. Statistics 23, 367–383 (1952).

An incomplete block design with v treatments, each replicated r times in b blocks of size k is said to be group divisible (G.D.) if the treatments can be divided into m groups of n each so that treatments belonging to the same group occur together λ_1 times and treatments belonging to different groups occur together λ_2 times. The authors classify these designs into 3 classes: singular G.D. designs $r=\lambda_1$; semiregular G.D. designs $r>\lambda_1$, $rk=v\lambda_2$; regular G.D. designs $r>\lambda_1$, $rk>\lambda_2v$. Various equations and inequalities, too numerous to report in a review, are derived for the parameters $r, b, k, m, n, \lambda_1, \lambda_2$. A singular G.D. design is equivalent to an incomplete balanced block design in which each treatment is replaced by a set of n treatments. For a semiregular G.D. design $k=cm$ and every block must contain c treatments from every group. By means of the theory of Hasse-Minkowski invariants the authors are able to derive strong conditions for the existence of a regular symmetrical G.D. design. The simplest of these is that $(r-\lambda_1)^{m(n-1)}(r^2-v\lambda_2)^{m-1}$ must be a perfect square. Other conditions are formulated in terms of the Hilbert norm-residue symbol and, although quite workable, are too complicated to report in a review. The authors add an impressive list of parameter combinations, meeting all the simpler necessary conditions for a

symmetrical G.D. design for which their theorems supply non-existence proofs. *H. B. Mann* (Columbus, Ohio).

Bush, K. A. A generalization of a theorem due to MacNeish. Ann. Math. Statistics **23**, 293–295 (1952).

A $k\times N$ matrix A whose elements a_{ij} belong to a finite set Σ containing s elements is said to be an orthogonal array (N, k, s, t) of size N, k constraints, s levels, and strength t if every $t\times N$ submatrix of A ($t\leqq k$) contains all the s^t possible column vectors with the same frequency λ ($N=\lambda s^t$). An orthogonal array with $t=2$, $\lambda=1$ is abstractly identical with a set of mutually orthogonal Latin squares with k constraints [Radhakrishna Rao, Proc. Edinburgh Math. Soc. 8, 119–125 (1949); these Rev. **11**, 710]. Denoting by $f(N, s, t)$ the maximum possible number of constraints for the orthogonal array (N, k, s, t) the author proves the following: If N_i is divisible by s_i^t for $i=1, 2, \cdots, u$, then

$$f(N_1N_2\cdots N_u, s_1s_2\cdots s_u, t)\geqq \min (k_1, k_2, \cdots, k_u)$$

where $k_i=f(N_i, s_i, t)$. This generalizes a theorem of MacNeish [Ann. of Math. **23**, 221–227 (1923)] which states that if $s=p_1^{n_1}p_2^{n_2}\cdots p_u^{n_u}$ where p_i are primes, then we can construct a set of mutually orthogonal Latin squares of k constraints where

$$k=\min (k_1, k_2, \cdots, k_u); \quad k_i=1+p_i^{n_i}=f(p_i^{2n_i}, p_i, 2).$$

R. C. Bose (Chapel Hill, N. C.).

Bush, K. A. Orthogonal arrays of index unity. Ann. Math. Statistics **23**, 426–434 (1952).

An orthogonal array $[N, k, s, t]$ of size N, k constraints, s levels, strength t, and index λ is a matrix (a_{ij}) of k rows and n columns whose elements are the integers $0, 1, \cdots, s-1$, $s>1$ with the property that the columns of any t-rowed submatrix contain each of the s^t possible t-rowed columns that can be formed with the numbers $0, 1, \cdots, s-1$ exactly λ times. The author considers only the case $\lambda=1$. If $f(N, s, t)$ denotes the maximum value of k for which $[N, k, s, t]$ exists, the author proves: If $s\leqq t$, then $f(s^t, s, t)\leqq t+1$. If $t<s$, $f(s^t, s, t)\leqq s+t-1$ if s is even and $\leqq s+t-2$ if s is odd. If $t<s$ and s is the power of a prime, the author constructs an $[s^t, s+1, s, t]$ by considering all s^t polynomials $y_1(x), \cdots, y_N(x)$ of degree $t-1$ in $GF(s)$ and putting $a_{ij}=y_j(e_i)$ ($i=1, \cdots, s$; $j=1, \cdots, s^t$), where $e_1, e_2, \cdots, e_s$ are the elements of $GF(s)$, the $(s+1)$th row is then formed by the leading coefficients of $y_j(x)$. That this is an orthogonal array follows from the fact that a polynomial of degree h over a field cannot have more than h roots. In the case $t=3$, $s=2^n$ yet another row may be added consisting of the second coefficient of $y_j(x)$. This is possible because the polynomial ax^2-b cannot have 2 distinct roots in a field of characteristic 2. If $s\leqq t$, the author constructs an orthogonal array of strength $t+1$ by taking as the first t rows the coefficients of the polynomial $y_j(x)$ and as the last row the element $y_j(1)$. Thus $f(s^t, s, t)=t+1$ if $s\leqq t$ and s is a prime power.

H. B. Mann (Columbus, Ohio).

Bulgakov, B. V. Division of rectangular matrices. Doklady Akad. Nauk SSSR (N.S.) **85**, 21–23 (1952). (Russian)

Let B, C be rectangular matrices. A necessary condition that X exist so that $XB=C$ hold is known to be: (right column-rank of C)$\leqq$(right column-rank of B). The full necessary and sufficient condition is that whenever a certain set of columns of B is right-linearly dependent, the corresponding set of columns of C (i.e., the set of columns with the same indices) be also linearly dependent with the same coefficients of linear dependence. Using this criterion, all divisors of C can be found. *J. L. Brenner.*

Parker, W. V., and Mitchell, B. E. Elementary divisors of certain matrices. Duke Math. J. **19**, 483–485 (1952).

Let P and Q be square matrices, and $f(x)$ and $g(x)$ polynomials such that $(P-Q)f(P)=(P-Q)g(Q)=0$. Then P and Q have the same elementary divisors except for those which are associated with characteristic roots which are roots of either $f(x)=0$ or $g(x)=0$. This generalizes recent results of Flanders [Proc. Amer. Math. Soc. **2**, 871–874 (1951); these Rev. **13**, 425]. *N. H. McCoy.*

Peremans, W., Duparc, H. J. A., and Lekkerkerker, C. G. A property of positive matrices. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 24–27 (1952).

Three proofs are given of the following theorem: Let the symmetric matrix C of order n have elements $c_{jk}=a_{jk}+ib_{jk}$, with a_{jk}, b_{jk} real. Suppose $\sum_{j,k=1}^n a_{jk}x_jx_k\geqq 0$ for all real x_i. If $\det c_{jk}=0$, then there exist real λ_i, not all zero, with $\sum_{k=1}^n c_{jk}\lambda_k=0$ (all j). Changed to matrix notation (all quantities real), the second proof establishes successively that: $(A+iB)(\lambda+\mu i)=0$; $\lambda'A\lambda=\lambda'B\mu=\mu'B\lambda=-\mu'A\mu=0$; $A\lambda=B\mu=A\mu=B\lambda=0$; $(A+iB)\lambda=0$. *G. E. Forsythe.*

Debreu, Gerard. Definite and semidefinite quadratic forms. Econometrica **20**, 295–300 (1952).

Let A be an $n\times n$ symmetric matrix; let B have n rows and $m\leqq n$ columns; x is an n-rowed column vector. The author gives a concise unified treatment, with original proofs, of known necessary and sufficient conditions for the quadratic form $x'Ax$ to be positive definite or semidefinite when x is subjected to the linear constraints $B'x=0$. In part, the proofs use the family of forms $x'Ax+\lambda x'BB'x$ and a lemma that

$$|A+\lambda BB'|=(-1)^m\begin{vmatrix} A & B \\ B' & 0_m \end{vmatrix}\lambda^m+\text{lower powers of }\lambda.$$

There is a brief historical survey. *G. E. Forsythe.*

Ostrowski, Alexandre. Sur les conditions générales pour la régularité des matrices. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **10**, 156–168 (1951).

Some of the author's known conditions and a new one for the regularity (non-singularity) of a matrix are proved to be the best of their types in a certain sense. If $A=(a_{\mu\nu})$ ($\mu, \nu=1, \cdots, n$), define $L_\mu^{(p)}=[\sum_{\nu\neq\mu}|a_{\mu\nu}|^p]^{1/p}$ and $m_\mu=\max_{i\neq\mu}|a_{\mu i}|$. Let $\alpha_\nu>0$ and $\lambda_\nu\geqq 0$ be given ($\nu=1, \cdots, n$). The theorems: I) Necessary and sufficient that each A with $|a_{\nu\nu}|\geqq\alpha_\nu$, $L_\nu^{(1)}\leqq\lambda_\nu$ (all ν) be regular is that for all ν_1, ν_2 with $\nu_1\neq\nu_2$ one have $\alpha_{\nu_1}\alpha_{\nu_2}>\lambda_{\nu_1}\lambda_{\nu_2}$. II) Necessary and sufficient that each A with $|a_{\nu\nu}|\geqq\alpha_\nu$, $m_\nu\leqq\lambda_\nu$ (all ν) be regular is that one have $\sum_{\nu=1}^n\lambda_\nu(\alpha_\nu+\lambda_\nu)^{-1}<1$. IV) Suppose $m_{\mu\nu}\geqq 0$ ($\mu, \nu=1, \cdots, n$). Let the matrix M have elements m_{ii} ($i=j$) and $-m_{ij}$ ($i\neq j$). Necessary and sufficient that each A with $|a_{\nu\nu}|\geqq m_{\nu\nu}$ (all ν) and $|a_{\mu\nu}|\leqq m_{\mu\nu}$ (all μ, ν with $\mu\neq\nu$) be regular is that M be positive definite.

Gerschgorin's Theorem [Izvestiya Akad. Nauk SSSR (7) 1931, 749–754] is extended by Theorem III: For all $p>0$, $q>0$ with $p^{-1}+q^{-1}=1$, each fundamental root λ of A satisfies the inequality

$$\sum_{\mu=1}^{n}\{1+|\lambda-a_{\mu\mu}|^q[L_\mu^{(p)}]^{-q}\}^{-1}\geqq 1,$$

where a summand with $\lambda - a_{\nu\nu} = L_\nu^{(\nu)} = 0$ is given the value 1. There are several numerical examples. *G. E. Forsythe.*

Ostrowski, A. Bounds for the greatest latent root of a positive matrix. J. London Math. Soc. **27**, 253–256 (1952).

Let $A = (a_{\mu\nu})$ be an $n \times n$-matrix with arbitrary $a_{\mu\nu} \neq 0$. Let $R_\mu = \sum_{\nu=1}^{n} |a_{\mu\nu}|$; $R = \max R_\mu$; $r = \min R_\mu$; $\kappa = \min_{\mu,\nu} |a_{\mu\nu}|$; $\sigma = (r-\kappa)^{\frac{1}{2}}(R-\kappa)^{-\frac{1}{2}}$. It is proved that, if ω is the root of $|\lambda E - A| = 0$ of greatest modulus, $|\omega| \leqq R - (1-\sigma)\kappa$; if, moreover, all $a_{\mu\nu} > 0$, then also $\omega \geqq r + (\sigma^{-1} - 1)\kappa$. If x_ν is the fundamental vector corresponding to ω, and if all $a_{\mu\nu} > 0$, it is also shown that $\kappa(R - r + \kappa)^{-1} < (\min x_\nu)(\max x_\nu)^{-1} \leqq \sigma$, an improvement of the Perron-Frobenius theorem that all $x_\nu > 0$. Some further improvements of the three results are indicated. *G. E. Forsythe* (Los Angeles, Calif.).

Parodi, Maurice. Sur un théorème de M. Ostrowski. C. R. Acad. Sci. Paris **234**, 282–284 (1952).

Ostrowski has shown that if limits are given for the under diagonal elements and over diagonal elements of an nth order matrix, then the characteristic roots lie within circles with a specified radius around the diagonal elements. In this note the author shows that if any of these circles does not have a point in common with the others, it contains one and only one characteristic root. The discussion is based on introducing a multiplicative factor t on the subdiagonal elements which is used as a parameter varying from 0 to 1. *F. J. Murray* (New York, N. Y.).

Azbelev, N., and Vinograd, R. A process of successive approximations for finding characteristic values and characteristic vectors. Doklady Akad. Nauk SSSR (N.S.) **83**, 173–174 (1952). (Russian)

In n-dimensional real or complex space let A be a linear operator, symmetric or not. Starting from a vector x_0 a new algorithm is proposed. For $k = 0, 1, \cdots$, compute successively

$$\lambda_k = (Ax_k, x_k)/\|x_k\|^2; \quad B_k = A - \lambda_k E \ (E = \text{identity});$$
$$y_k = B_k^* B_k x_k; \quad \eta_k = (y_k, x_k)/\|y_k\|^2; \quad x_{k+1} = x_k - \eta_k y_k.$$

Three theorems are announced, including: (I) Either $x_k \to 0$ or all limit points of $\{x_k\}$ lie in a single invariant space of A and have the same norm $d > 0$; (II) If the eigenvalue λ' of A corresponds to an invariant space L' consisting only of eigenvectors, and if x_0 makes a sufficiently small angle with L', then there exists x' in L' such that $\|x_k - x'\| \leqq Cq^k$, $q < 1$. If A is spanned by its eigenvectors, the authors conjecture that the x_0 for which $x_k \to 0$ are nowhere dense. *G. E. Forsythe* (Los Angeles, Calif.).

Lepage, Th. Sur une classe de polynômes irréductibles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 412–425 (1952).

Let K be a field, let n be an integer not divisible by the characteristic of K, let $Z = (z_{ij})_{1 \leqq i \leqq n, 1 \leqq j \leqq n}$ be a symmetric matrix such that $(z_{ij})_{1 \leqq i \leqq j \leqq n}$ is a family of indeterminates. The author shows that every polynomial which is a linear combination over K of arbitrary minors of Z is absolutely irreducible. The vector space over K of all such polynomials is proved isomorphic to a certain subspace of the space of exterior forms of degree n of a vector space over K of dimension $2n$; the latter is invariant and irreducible with respect to the symplectic group. By means of this isomorphism certain problems concerning these polynomials are studied. *E. R. Kolchin* (New York, N. Y.).

Young, Alfred. On quantitative substitutional analysis. IX. Proc. London Math. Soc. (2) **54**, 219–253 (1952).

Alfred Young died in 1940, and the present paper was found, typed and filled in, along with a large body of mss. The Introduction by this reviewer gives a brief account of these mss.

In this paper, Q.S.A.IX, Young is concerned with his theory of reductibility as given in Q.S.A.VII [same Proc. **36**, 304–368 (1933)] and in making clear the extent to which reductibility differs from the familiar concept of reducibility. Part I extends the reduction formula for binary forms developed in Theorem II of Q.S.A.VII and its corollary (p. 313). In Part II these ideas are further developed and applied to invariants and covariants of orders 1, 2. Part III corrects the statements of two theorems in Q.S.A.VIII [ibid. **37**, 441–495 (1934)]. Parts IV, V, and VI contain essentially new material and show how the classical theory of canonical forms fits into and makes more precise the criteria of reductibility already obtained. In Part VII these criteria are analysed in the general case and in Part VIII they are applied to obtain a list of irreductible forms for the quintic. Since reducibility implies reductibility, this list includes all the irreducible forms and these are identified according to a standard notation. In the last sentence of the paper Young promises a discussion of reducibility, but this seems never to have been attempted. *G. de B. Robinson* (Toronto, Ont.).

Scarf, Herbert. Group invariant integration and the fundamental theorem of algebra. Proc. Nat. Acad. Sci. U. S. A. **38**, 439–440 (1952).

The fundamental theorem of algebra is deduced from the existence of Haar measure on a compact group of matrices. *L. H. Loomis* (Cambridge, Mass.)

Abstract Algebra

Nikodým, Otton Martin. Critical remarks on some basic notions in Boolean lattices. I. Anais Acad. Brasil. Ci. **24**, 113–136 (1952).

The non-equivalence of various ways of defining homomorphism for partially ordered systems, in particular for Boolean algebras, is shown by simple examples. (The postulate: aRa', $a = b$, bRb' implies $a' = 'b'$ should be added to the postulates listed in §12 to make §14 valid.) *I. Halperin.*

Fogelis, E. On the axioms of arithmetic. Latvijas PSR Zinātņu Akad. Vēstis **1949**, no. 3(20), 96–101 (1949). (Latvian. Russian summary)

Let K be a denumerable set in which two binary operations (called addition and multiplication) are defined. Hypotheses: 1) K is a commutative semi-group with respect to both operations and the distributive law holds. 2) There exists a subset $k \subset K$ such that every element in $K - k$ admits a unique representation as a sum of a finite number of elements of k. 3) There exist a subset $\pi \subset k$ such that every element of $k - \pi$ which is not a multiplicative unit admits a unique representation as a product of a finite number of elements of π. Conclusion: k consists of a single element. *L. Bers* (New York, N. Y.).

Lorenzen, Paul. Teilbarkeitstheorie in Bereichen. Math. Z. 55, 269–275 (1952).

This note, like two earlier papers by the same author [Math. Z. 45, 533–553 (1939); 52, 483–526 (1949); these Rev. 1, 101; 11, 497] is concerned with an abstract description of some features of multiplicative ideal theory. The present paper, like its predecessor, is ultimately based on the following idea. The relation of divisibility in the quotient field of a given integral domain defines a partial ordering of the elements of the multiplicative group G of the field. The corresponding ideals are represented by a particular extension of the partial ordering of G in which each pair of elements possesses a greatest common divisor (in the sense of partial ordering theory). The notion of integral dependence also can be replaced by an abstract definition within this framework.

In the preceding paper the author replaced G as described above by a partially ordered group. In the present paper he goes one step further and assumes that G is merely a partially ordered domain with operators which is included in a partial ordering for which each pair of elements possesses a greatest common divisor. It is shown that the main theorems concerning the representation of the partial ordering of G as a conjunction of admissible orderings remain applicable except for minor modifications. *A. Robinson.*

Curtis, Charles W. On additive ideal theory in general rings. Amer. J. Math. 74, 687–700 (1952).

For two ideals A, B of a general ring R the author denotes by AB^{-1} the ideal consisting of all $x\varepsilon R$ such that $xB\subseteq A$. The join S of all ideals C such that $A(C+B)^{-1}\neq A$ whenever $AB^{-1}\neq 0$ is called the adjoint ideal of A. An ideal A is called primal if for the join P of all ideals B such that $AB^{-1}\neq A$ one has again $AP^{-1}\neq A$. It follows that P is the adjoint ideal of A and that P is a prime ideal. In the commutative case any primal ideal is also primal in the sense of L. Fuchs [Proc. Amer. Math. Soc. 1, 1–6 (1950); these Rev. 11, 310] whereas, as is shown by an example, primality in the sense of Fuchs does not imply primality as defined by the author. An ideal A is called irreducible, respectively strongly irreducible, if A cannot be expressed as an intersection of a finite number, respectively any number (finite or infinite), of proper divisors of A. The author shows that every irreducible ideal is primal and that every ideal is the intersection of primal ideals. In the presence of the ascending chain condition (in short, A.C.C.) for ideals it follows that every ideal can be represented as an intersection of a finite number of primal ideals. A representation of this type is called normal if it is irredundant and if no component can be replaced by a proper divisor. If the A.C.C. holds, every ideal has a normal representation by primal ideals such that in the set of the corresponding adjoint primes no ideal divides another. The author succeeds in showing that in any two representations of this kind the number of components and the corresponding adjoint ideals are the same. An ideal P is said to be a maximal prime ideal of A if $P\supseteq A$, $AP^{-1}\neq A$ and if for any proper divisor Q of P one has $AQ^{-1}=A$. It is shown that the adjoint ideal of A is the intersection of all maximal ideals of A, and hence by a result of McCoy [Amer. J. Math. 71, 823–833 (1949); these Rev. 11, 311] is a divisor of the McCoy radical of A. It is shown that an ideal P is a maximal prime ideal belonging to A if and only if P is one of the adjoint primes in a normal representation of A by primal ideals. Turning to the study of primary ideals in rings satisfying the A.C.C. for ideals the author assigns in the usual manner to any primary ideal Q the prime ideal P belonging to Q and remarks that Q is primal with the adjoint prime P. The converse need not be true, as is shown by an example due to Fuchs [loc. cit.]. The author then proceeds to study shortest representations by primary ideals and also obtains some results on the uniqueness of isolated component ideals. The first part of the paper concludes with a result concerning the Jacobson radical J of a ring R having a Noetherian ideal theory, i.e., satisfying the A.C.C. for right ideals and having the property that every ideal in R is an intersection of a finite number of primary ideals. It is shown that in such rings one has the equality $J^\omega=0$.

In the second part of the paper various examples of rings with a Noetherian ideal theory are discussed and the following result is proved: Let S be a simple ring with unit element and φ the center of S. Denote by $S[x]$, respectively $S\{x\}$, the algebra over φ of polynomials, respectively of formal power series, in n indeterminates x_i with coefficients in S, where it is assumed that the x_i are commutative with each other and with the elements of S. If either $R=S[x]$ (for an arbitrary S) or $R=S\{x\}$ and $\{S:\varphi\}<\infty$, then R has a Noetherian ideal theory. *J. Levitzki* (Jerusalem).

Rédei, L. Die Vollidealringe. Monatsh. Math. 56, 89–95 (1952).

The ring R is a vollidealring (v-ring) if every subring of R is an ideal of R. For any integers d and d_1, let $R(d, d_1)=I_d[x]/(x^2-d_1x)$, where I_d is the ring of integers modulo d. This paper proves that a v-ring is one of the following: I) A zero ring; II) a direct sum of a $R(0, d_1)$ $(d_1>0)$ and a zero ring R with $d_1R=0$; III) a direct sum $\sum_p R_p$, p prime, where R_p is either a zero-ring having all its elements of characteristic a power of p or a direct sum of a ring $R(p^e, p^f)$ $(0\leqq f<e)$ and a zero ring R_p' with $p^fR_p'=0$. *R. E. Johnson* (Northampton, Mass.).

Rosenberg, Alex. Subrings of simple rings with minimal ideals. Trans. Amer. Math. Soc. 73, 115–138 (1952).

This paper is concerned with extending certain classical results for finite-dimensional central simple algebras to simple rings with minimal one-sided ideals (S.M.I. rings). Recall that an S.M.I. ring is of the form $\mathfrak{F}(\mathfrak{M}, \mathfrak{N})$, where $\mathfrak{M}$, $\mathfrak{N}$ are dual spaces over a division ring $\mathfrak{D}$ and $\mathfrak{F}(\mathfrak{M}, \mathfrak{N})$ is the ring of all linear transformations T on $\mathfrak{M}$ having finite rank and possessing adjoints T^* on $\mathfrak{N}$ [Dieudonné, Bull. Soc. Math. France 70, 46–75 (1942); these Rev. 6, 144; Jacobson, same Trans. 57, 228–245 (1945); these Rev. 6, 200]. A ring is called a Zorn ring if every non-nil ideal contains a non-zero idempotent. It is then shown that any simple Zorn subring of an S.M.I. ring is itself an S.M.I. ring. If B is a simple Zorn subring of an S.M.I. ring A, then the "commutator" of B in A is defined as the set $A(B)$ of all elements of A which commute with each element of B. It is proved that, if B does not satisfy the descending chain condition (d.c.c.), then $A(B)$ is equal to the two-sided annihilator of B in A. If A is an algebraic algebra over an algebraically closed field Φ and B satisfies the d.c.c. with unit u, then (i) $A(B)$ is the direct sum of the commutator of B in uAu and the two-sided annihilator of B in A; (ii) if A has infinite dimension over Φ, $A(A(B))=B$; (iii) $A(A(B))=B\oplus Z$, $Z\cong\Phi$.

Again let B be a simple Zorn subring of an S.M.I. ring $A=\mathfrak{F}(\mathfrak{M}, \mathfrak{N})$ and denote by B^* the ring of adjoints of elements of B. Let $\mathfrak{T}(B)$ be the subspace in $\mathfrak{M}$ of all vectors annihilated by each element of B. Then B is said to be

"caudal" or "acaudal" in A according as $\mathfrak{M}B\oplus\mathfrak{T}(B)\neq\mathfrak{M}$ or $\mathfrak{M}B\oplus\mathfrak{T}(B)=\mathfrak{M}$. It is then shown that, if B does not satisfy the d.c.c., then $A(B)$ is a ring with a nilpotent radical R, $R^2=0$, and $A(B)-R$ is an S.M.I. ring. If B is acaudal in A and B^* is acaudal in A^*, then $A(B)$ is an S.M.I. ring isomorphic to $\mathfrak{F}(\mathfrak{T}(B), \mathfrak{T}(B^*))$. The acaudal case can also be generalized to the case in which A is the algebra of all linear transformations (not necessarily of finite rank) on a vector space $\mathfrak{M}$ over a field.

The next problem concerns extension of isomorphisms between two simple subrings of a S.M.I. ring $A=\mathfrak{F}(\mathfrak{M}, \mathfrak{N})$, where $\mathfrak{M}$, $\mathfrak{N}$ are dual spaces over an algebraically closed field Φ. If B is a simple subring of A, then $\mathfrak{M}B$ is a direct sum $\sum U_i$ of irreducible B-modules, the cardinal number of the index set being uniquely determined and called the "height" of B in A [Dieudonné, Comment. Math. Helv. 21, 154–184 (1948); these Rev. 9, 563]. The main result here is the following. Let B, C be isomorphic simple subalgebras of A. Then the isomorphism between B and C can be extended to an inner automorphism of $\mathfrak{L}(\mathfrak{M}, \mathfrak{N})$ (the ring of all linear transformations on $\mathfrak{M}$ with adjoints on $\mathfrak{N}$) if (i) $A(B)$ and $A(C)$ are isomorphic; (ii) the heights of B and C in A are equal; (iii) B and C are acaudal in A and B^* and C^* are acaudal in A^*. (Examples show that both acaudal conditions in (iii) are needed.)

Let $A=\mathfrak{F}(\mathfrak{M}, \mathfrak{N})$, where $\mathfrak{M}$, $\mathfrak{N}$ are dual spaces of countably infinite dimensions over the division ring $\mathfrak{D}$. Two subrings B, C of A are said to be "equivalent" if there exists $P\in\mathfrak{L}(\mathfrak{M}, \mathfrak{N})$ such that $P^{-1}BP=C$. Let $\mathfrak{R}$ be a subspace of $\mathfrak{M}$ and denote its "orthogonal complement" in $\mathfrak{N}$ by $\mathfrak{R}'$. The three cardinals, $\dim\mathfrak{R}$, $\dim\mathfrak{R}'$ and $\dim(\mathfrak{R}''/\mathfrak{R})$ are called the invariants of $\mathfrak{R}$ in $\mathfrak{M}$. The following results are then obtained: Two left ideals in A are equivalent if and only if their ranges in $\mathfrak{M}$ have the same invariants. Let B, C be two S.M.I. subrings of $\mathfrak{F}(\mathfrak{M}, \mathfrak{N})$, which induce dense rings of linear transformations on their respective ranges, and assume that $\mathfrak{T}(B)=\mathfrak{T}(B^*)=\mathfrak{T}(C)=\mathfrak{T}(C^*)=0$. Then B and C are equivalent if and only if the invariants of $\mathfrak{M}B$, $\mathfrak{M}C$ in $\mathfrak{M}$ are equal and the invariants of $B^*\mathfrak{N}$, $C^*\mathfrak{N}$ in $\mathfrak{N}$ are equal. *C. E. Rickart* (New Haven, Conn.).

Nollet, Louis. Sur les anneaux premiers. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 287–294 (1952).

A two-sided ideal I in a heterogeneous ring R [see Apéry, Colloques Internationaux du Centre National de la Recherche Scientific, no. 24, Paris, 1950, pp. 107–108; these Rev. 13, 100] is called prime if xy in I implies that x is in I or y is in I. The ring R is a prime ring if every two-sided ideal is prime. A heterogeneous field is defined, and it is proved that every prime commutative ring is a field.

F. Kiokemeister (South Hadley, Mass.).

Kaplansky, Irving, and Mackey, George W. A generalization of Ulm's theorem. Summa Brasil. Math. 2, 195–202 (1951).

Ulm's theorem gives a complete classification of countable Abelian torsion-groups by means of invariants [Ulm, Math. Ann. 107, 774–803 (1933); Zippin, Ann. of Math. 36, 86–99 (1935); Kuroš, Teoriya grupp, Moscow-Leningrad, 1944, chap. VIII; these Rev. 9, 267]. The generalization to countably generated modules over any principal ideal ring (in place of the ring of integers) is almost trivial. The generalization contained in the present paper is far from trivial and goes in the following direction: There is a restriction on the ring over which the modules are to be considered. It is not an arbitrary principal ideal ring, but a complete discrete valuation ring (e.g., the p-adic integers). On the other hand there is more freedom in the choice of the modules. They need not be torsion-modules, but can have a torsion-free part of rank 1. Let R be a complete discrete valuation ring belonging to the prime p, and M a countably generated R-module. The elements of M which are annihilated by some power of p form a submodule T, the torsion module of M. The residue-class module M/T which is torsion-free is assumed to be of rank 1. M is assumed to be reduced, i.e., it contains no completely divisible submodule N (with the property $pN=N$). Define modules M_α by $M_0=M$, $M_{\alpha+1}=pM_\alpha$, $M_\alpha=\bigcap_{\beta<\alpha}M_\beta$ for limit ordinals α. This decreasing chain terminates in 0 with an index λ, the "type" of M. The height $h(x)$ of an element x of M which plays the most important part in the discussion is defined as that index α for which $x\in M_\alpha$, x non-$\in M_{\alpha+1}$. Let P be the module of all elements x with $px=0$. Write $P_\alpha=P\cap M_\alpha$. Then $P_\alpha/P_{\alpha+1}$ can be considered as a vector space over the residue class field $R/(p)$. Its dimension is denoted by $f(\alpha)$ and the set of cardinals $f(\alpha)$ are the Ulm invariants of M. Lastly let x be an element of infinite order in M. We form the sequence of ordinals $g(i)=h(p^ix)$; different elements of infinite order may give rise to different sequences $g(i)$, but since any two of them, say x and x', are related in the form $x=p^nx'+y$ with y of finite order, the sequences become identical after omission of suitable initial segments from each. This leads naturally to an equivalence class of sequences $g(i)$ which is now an invariant of M. The authors' theorem is then: The Ulm invariants and the equivalence class of $g(i)$ form a complete set of invariants for M. Incidentally the paper yields a new proof of Ulm's original theorem if certain simplifications are taken into account. In that form it is the shortest and the most lucid proof. It has found its way into the latest Bourbaki [Eléments de mathématique, XIV, pp. 79–81, Actualités Sci. Ind. no. 1179, Hermann, Paris, 1952].

K. A. Hirsch (London).

Samuel, P. Some asymptotic properties of powers of ideals. Ann. of Math. (2) 56, 11–21 (1952).

Let $\mathfrak{A}$ be a commutative Noetherian ring with unit. Let A and B be two ideals of S satisfying: (1) A and B have the same radical R; (2) R is not nilpotent; (3) $\bigcap_{i=1}^{\infty}R^i=(0)$. Then for each positive integer n there exists a largest integer m such that $A^n\subset B^m$ and a smallest integer q such that $A^n\supset B^q$. Denote these two integers by $v_B(A, n)$ and $w_B(A, n)$ respectively. Denote by $l_B(A)$ (resp. $L_B(A)$) the least upper (resp. greatest lower) bound of the sequence $\{v_B(A, n)/n\}$ (resp. $\{w_B(A, n)/n\}$). It is proved that the sequences actually converge to these respective bounds. Various elementary consequences of the definition are obtained, including $0<l_B(A)\leqq L_B(A)<\infty$, and $L_B(A)l_A(B)=1$. The relation $l_B(A)=L_B(A)=1$ defines an equivalence relation with which addition and multiplication of ideals are compatible. The set $I(\mathfrak{A})$ of equivalence classes is thus a system with addition and multiplication. If R is a given ideal, the subset $I_R(\mathfrak{A})$ consisting of the classes of ideals with R as radical is closed under the operations in $I(\mathfrak{A})$. Multiplication in $I_R(\mathfrak{A})$ satisfies the cancellation law.

The relation $l_B(A)\geqq 1$ induces in $I(\mathfrak{A})$ an order relation which is compatible with multiplication, and with respect to which $I(\mathfrak{A})$ is a lattice. If s is a positive real number, the relation $l_B(A)=L_B(A)=s$ defines a relation between the classes α and β of A and B in $I(\mathfrak{A})$, and one writes $\alpha=\beta^s$. This exponentiation, when defined, is unique and satisfies

the usual elementary rules. It also satisfies: (*) If α and β are in $I_B(\mathfrak{A})$ and if α is not a power of β, then $\alpha^x\beta^y=(\alpha^u\beta^v)^w$ implies $x=uw$, $y=vw$. With the aid of this theorem a notion of straight line and a barycentric calculus can be introduced into the set of equivalence classes of $I_B(\mathfrak{A})$ with respect to the relation "α is a power of β". A homomorphism of $\mathfrak{A}$ induces an order-preserving homomorphism of $I(\mathfrak{A})$.

If $\mathfrak{A}$ is a local ring of dimension $d\geqq 1$, and if R is its maximal ideal, then

$$l_B(A)\leqq(e(A)/e(B))^{1/d}\leqq L_B(A),$$

where $e(A)$ is the multiplicity previously defined by the author [J. Math. Pures Appl. (9) 30, 159–205 (1951); these Rev. 13, 980]. If $\mathfrak{A}$ is a local domain of dimension 1 whose integral closure is a finite $\mathfrak{A}$-module, and if $\mathfrak{A}/R$ is infinite, then every ideal in $\mathfrak{A}$ is equivalent to a principal ideal. Moreover $\mathfrak{A}x$ and $\mathfrak{A}y$ are equivalent if and only if $w_i(x)=w_i(y)$, where the w_i are the valuations of the quotient field of $\mathfrak{A}$ whose v-rings contain $\mathfrak{A}$.

The proofs in this paper follow fairly easily from the definitions, the cancellation law in $I(\mathfrak{A})$, and (*). The latter two follow from: Let A, B be as above and in addition let $A\subset B$. Then there exist integers c and s and an element $A\in A^s$ such that for all integers $n\geqq c+s$ and p,

$$(A^nB^p:Aa)\cap A^c=A^{n-s}B^p.$$

This generalizes a previous result of the author [loc. cit., remark on p. 182]. The proof of this last result, which is not easy, involves an extension of the methods developed by the author in the earlier paper. *I. S. Cohen.*

Nakayama, Tadasi. Note on double-modules over arbitrary rings. Amer. J. Math. 74, 645–655 (1952).

Soit A un anneau (commutatif ou non) ayant un élément unité, M un bimodule (à droite et à guache) unitaire sur A, ayant la propriété qu'il existe un élément $u_0\in M$ tel que M, considéré comme A-module à gauche, admette une base formée d'éléments de $u_0\cdot A$. Soit M^* le dual de M quand M est considéré comme A-module à gauche, et pour tout $\sigma\in M^*$, soit $\bar\sigma$ l'endomorphisme du groupe additif A défini par $\bar\sigma(x)=\sigma(u_0x)$ pour $x\in A$; l'ensemble $R(M,u_0)$ des $\bar\sigma$ est un sous-groupe additif de l'anneau $\mathfrak{A}$ des endomorphismes du groupe additif A. L'auteur obtient la caractérisation des bimodules M du type précédent qui sont isomorphes à un produit tensoriel $A\otimes_S A$ de A par lui-même, relativement à un sous-anneau S de A (contenant l'élément unité 1 de A), cet isomorphisme devant appliquer u_0 sur $1\otimes 1$: il faut et il suffit que $R(M,u_0)$ soit un sous-anneau de $\mathfrak{A}$, et que la relation $u_0x=0$ ($x\in A$) implique $x=0$. Ceci généralise un résultat de Hochschild [même J. 71, 443–460 (1949); ces Rev. 10, 676] et du rapporteur [ibid. 73, 13–19 (1951); ces Rev. 12, 476] qui correspond au cas particulier où A est un corps (commutatif ou non). *J. Dieudonné.*

Kleinfeld, Erwin. An extension of the theorem on alternative division rings. Proc. Amer. Math. Soc. 3, 348–351 (1952).

A new result is added to the recent sequence of theorems on alternative rings. The hypothesis is that R is a simple alternative ring without nilpotent elements, and that R does not have characteristic two. Then it follows that R is either associative or is a Cayley-Dickson division algebra over its center. *Marshall Hall* (Columbus, Ohio).

Jaeger, Arno. Adjunction of subfield closures to ordered division rings. Trans. Amer. Math. Soc. 73, 35–39 (1952).

B. H. Neumann [dieselben Trans. 66, 202–252 (1949); diese Rev. 11, 311] hat bewiesen, dass ein geordneter Divisionsring Σ mit dem Zentrum Z und dem zum Körper der rationalen Zahlen anordnungsisomorphen Primkörper $P\leqq Z$ zu einem geordneten Divisionsring $\Sigma^*\geqq\Sigma$ erweitert werden kann, der in seinem Zentrum einen Teilkörper P^* enthält, der anordnungsisomorph ist zum Körper der reellen Zahlen. Hier wird mit einer Modifizierung der Neumannschen Methoden der schärfere Satz bewiesen: Ist F ein beliebiger Unterkörper von Z, so kann der geordnete Divisionsring Σ zu einem geordneten Divisionsring Σ^* erweitert werden, der die Anordnung von Σ erhält und in seinem Zentrum einen Körper enthält, der anordnungsisomorph ist zur abgeschlossenen Hülle F^* von F, die aus der Anordnung von F topologisch definiert ist und mit Hilfe von Fundamentalfolgen und Nullfolgen in der üblichen Weise konstruiert werden kann [siehe auch L. W. Cohen und C. Goffman, ibid. 67, 310–319 (1949); diese Rev. 11, 324]. *R. Moufang* (Frankfurt a.M.).

Goldie, A. W. The scope of the Jordan-Hölder theorem in abstract algebra. Proc. London Math. Soc. (3) 2, 349–368 (1952).

Given an algebra A and fixed subalgebra A_0, consider only subalgebras containing A_0 and congruences $\mathfrak{R}_i$ whose domain $D(\mathfrak{R}_i)$ contains A_0. Denote by $\mathfrak{U}_B$ the relation under which all elements of B are congruent, by $\{\mathfrak{R}\}$ the set of all $a\in A$ congruent under $\mathfrak{R}$ to elements of A_0, by $[B]\mathfrak{R}$ the set of all $x\in A$ with $x\mathfrak{R}b$ for some $b\in B$. For fixed $\mathfrak{R}$, $\mathfrak{S}$, set $E=D(\mathfrak{R})\cap D(\mathfrak{S})$, $\mathfrak{R}'=\mathfrak{R}\cap\mathfrak{U}_E$, $\mathfrak{S}'=\mathfrak{S}\cap\mathfrak{U}_E$; $\mathfrak{R}$, $\mathfrak{S}$ are "weakly permutable" if $[\{\mathfrak{R}'\}]\mathfrak{S}'=[\{\mathfrak{S}'\}]\mathfrak{R}'$. A subalgebra C of $B\subset A$ is "normal in B" if for some $\mathfrak{R}$, $D(\mathfrak{R})=B$ and $\{R\}=C$; such an $\mathfrak{R}$ is denoted $\mathfrak{R}(B,C)$. A congruence is an "upper congruence" if it is the greatest (least refined) congruence in the set of all $\mathfrak{R}(B,C)$ for some B, C. A series $\mathfrak{R}_1,\mathfrak{R}_2,\cdots,\mathfrak{R}_{n+1}$ of congruences is a "normal series of congruences for $A_1\subset A$" if $\{\mathfrak{R}_1\}=A_1$, $\{\mathfrak{R}_{n+1}\}=A_0$, $D(\mathfrak{R}_1)=A_1$, $D(\mathfrak{R}_{i+1})=\{\mathfrak{R}_i\}$. It is a "maximal series for A_1" if in addition each $\mathfrak{R}_i$ is an upper congruence with $D(\mathfrak{R})=B$, not properly contained in any other such upper congruence except $\mathfrak{U}_B$. Generalizing the Jordan-Hölder theorem, if A satisfies ascending and descending chain conditions, then A satisfies the Jordan-Hölder condition (of one-to-one correspondence of isomorphic factors) for maximal series of congruences if and only if the terms $\mathfrak{R}_i$, $\mathfrak{S}_j$ of any two maximal series for any subalgebra A_1 are pairwise weakly permutable. Zassenhaus refinements are similarly considered. The usual theorem on lengths of chains in modular lattices is derived from the main generalization. *P. M. Whitman.*

Kuročkin, V. M. The decomposition of algebras into a semi-direct sum of a radical and a semi-simple subalgebra. Moskov. Gos. Univ. Učenye Zapiski 148, Matematika 4, 192–203 (1951). (Russian)

Let A be an algebra, R its Jacobson radical, $B=A/R$. The problem is to find a subalgebra S of A such that A is the vector-space direct sum of S and R. The author gives two cases where the answer is affirmative: (1) R finite-dimensional, B separable and such that every ideal of finite index in B has a unit element; (2) R locally nilpotent, B of countable dimension and locally separable (every finite set is contained in a finite-dimensional separable subalgebra). The proofs proceed by reducing the problem to the finite-

dimensional case. Two negative examples are offered. In the first A/R is one-dimensional, and R is a suitable radical algebra with no divisors of zero; the second example shows that the countability assumption in the second theorem cannot be dropped. *I. Kaplansky* (Chicago, Ill.).

Fell, J. M. G., and Tarski, Alfred. On algebras whose factor algebras are Boolean. Pacific J. Math. **2**, 297–318 (1952).

Uniqueness of direct decomposition is investigated for algebras A possessing a binary operation with neutral element 0. A necessary, and in the finite case sufficient, condition is that, whenever B and the direct product $C_1\times C_2$ are direct factors of A, then

$$(\Delta)\qquad B\cap(C_1\times C_2)=(B\cap C_1)\times(B\cap C_2).$$

This is shown equivalent to the condition that

$$A=B\times C=B\times D$$

imply $C=D$. Generally, the solutions of $B\times C=B\times D$ are determined by the homomorphisms of B into the 'center' of C, that is, the abelian group of those elements that have inverses and that commute and associate with all elements of C. If the only homomorphism of A into its center is trivial, condition Δ holds. This condition is established (directly) for generalized (=locally) cyclic groups, and, using the above, for groups with trivial center, for perfect groups, and for rings with unit. *R. C. Lyndon* (Princeton, N. J.).

Cohn, P. M. A theorem on the structure of tensor-spaces. Ann. of Math. (2) **56**, 254–268 (1952).

If $\mathfrak{A}=\mathfrak{F}[x_1,\cdots,x_q]$ is a free associative algebra with q generators over a field $\mathfrak{F}$ of characteristic zero, then $\mathfrak{A}$ as a vector space over $\mathfrak{F}$ has the direct decomposition $\mathfrak{A}=U\oplus U^2\oplus\cdots$, where U^n is the space of all homogeneous forms of degree n in the x's. This paper is a study of the structure of the tensor-space U^n. If $\mathfrak{g}$ is the group of nonsingular linear transformations of U and if $\mathfrak{v}\subset\mathfrak{g}$ is the group of permutations of the set $(x_1,\cdots,x_q)$, then let $\mathfrak{G}$ and $\mathfrak{B}$ be the rings of linear transformations of U^n induced by $\mathfrak{g}$ and $\mathfrak{v}$ respectively. The elements of the symmetric group S_n also induce linear transformations of U^n: for $x_{i_1}\cdots x_{i_n}\in U^n$, $\gamma\in S_n$, define $(x_{i_1}\cdots x_{i_n})\gamma=x_{j_1}\cdots x_{j_n}$, where $i_{k\gamma}=j_k$, and extend γ to U^n by linearity. Denote by $\mathfrak{P}$ the ring of linear transformations of U^n so generated by S_n. Let $\alpha=(\alpha_1,\cdots,\alpha_q)$, where $\sum\alpha_i=n$, and let Z^α be the subspace of U^n consisting of all forms homogeneous of degree α_i in x_i, $i=1,\cdots,q$. The subgroup of $\mathfrak{v}$ leaving invariant Z^α is denoted by $\mathfrak{v}^\alpha$. The principal theorem states that $\sum Z^\alpha$ is a direct decomposition of U^n into $\mathfrak{P}$-modules, and there exist $\theta_{\alpha\beta}\in\mathfrak{G}$ such that $\theta_{\alpha\beta}$ is a $\mathfrak{P}$-homomorphism of Z^α into Z^β. Furthermore, if V is any $\mathfrak{g}$-submodule of U^n, then $V^\alpha=V\cap Z^\alpha$ is a $\mathfrak{v}^\alpha$-module, $V=\sum V^\alpha$, and the mapping $W\to W\theta_{\alpha\alpha}$ is a join-homomorphism of the lattice of $\mathfrak{g}$-submodules of V onto the lattice of $\mathfrak{v}^\alpha$-submodules of V^α which is an isomorphism in case $V=U^n$ and $q\geqq n$. *R. E. Johnson.*

Seidenberg, A. Some basic theorems in differential algebra (characteristic p, arbitrary). Trans. Amer. Math. Soc. **73**, 174–190 (1952).

The author discusses anew several results concerning differential polynomials and differential fields of characteristic zero, first proved by J. F. Ritt and by some of his students, and extends them (in some cases in a reformulated version) to the case of nonzero characteristic. He also proves an analog of MacLane's theorem on the existence of a separating transcendence base. The considerations are all restricted to ordinary differential fields. *E. R. Kolchin.*

Jaeger, Arno. Eine algebraische Theorie vertauschbarer Differentiationen für Körper beliebiger Charakteristik. J. Reine Angew. Math. **190**, 1–21 (1952).

Because of the complexity of notation it must suffice to indicate the results of this paper as follows. Suppose that $K=\{z\}$ is a field; then an isomorphism $\mathfrak{D}$ of K into the formal power series ring $K\{\mathfrak{u}\}=K\{u_1,\cdots,u_n\}$ of n indeterminates, $z\to z\mathfrak{D}(\mathfrak{u})$ is called a multidifferentiation of dimension n if $(\mathfrak{D}^{\mathfrak{n}}z)_{\mathfrak{D}}(\mathfrak{u})=\Delta^{\mathfrak{n}}(z\mathfrak{D}(\mathfrak{u}))$ where $\mathfrak{n}=(M_1,\cdots,M_n)$ is a vector of positive integers indicating formal partial differentiations (coefficient of $u_1^{M_1}\cdots u_n^{M_n}$), and where $\Delta^{\mathfrak{n}}$ stands for the ordinary (Taylor coefficients) in the power series expansion. The author extends to such general differentiations the results of Schmidt and Hasse [same J. **177**, 215–237 (1937)] which form the 1-dimensional case. Of special interest are the regular differentiations for which there exists an $\mathfrak{n}$-tuple of elements $\mathfrak{x}$ in K such that det $(\mathfrak{D}\mathfrak{x})\neq0$. After having generalized to partial differential equations (characteristic $p\neq0$) the above mentioned results for dimension 1 (§5), it is shown that a regular multidifferentiation can be approximated by sequences of differentiations with respect to systems of n-tuples of elements in K (Theorem 7). Next, two multiple differentiations $\mathfrak{A}$, $\mathfrak{B}$ of respective dimensions m, n are called permutable if $z\to z\mathfrak{A}(\mathfrak{u})'\mathfrak{B}(\mathfrak{v})=z\mathfrak{B}(\mathfrak{v})'\mathfrak{A}(\mathfrak{u})$ is a multidifferentiation of dimension $m+n$, where the prime indicates that $\mathfrak{B}$, $\mathfrak{A}$ are trivially applied to the variables $\mathfrak{u}=(u_1,\cdots,u_m)$, $\mathfrak{v}=(v_1,\cdots,v_n)$, respectively. The author then establishes criteria for the permutability of regular differentiations in terms of their corresponding approximating elementary differentiations (Theorems 8, 9, 10). Finally the results are applied to algebraic function fields of dimension n where it is shown (extending the results on derivations) that $n+1$ multidifferentiations of dimension 1 are dependent (Theorem 11). *O. F. G. Schilling* (Chicago, Ill.).

Rose, I. H. On the cohomology theory for associative algebras. Amer. J. Math. **74**, 531–546 (1952).

L'auteur se place dans la théorie de la cohomologie des algèbres associatives, due à Hochschild [Ann. of Math. **46**, 58–67 (1945); **47**, 568–579 (1946); Duke Math. J. **14**, 921–948 (1947); ces Rev. **6**, 114; **8**, 64; **9**, 267]. Convenons de dire qu'une algèbre A est de dimension $\leqq m$ si $H^{m+1}(A, Q)=0$ pour tout A-bimodule Q; de dimension m si A est de dimension $\leqq m$ et n'est pas de dimension $\leqq m-1$. Etant données deux algèbres A et B, quelles sont les relations existant entre la dimension de A, celle de B et celle du produit tensoriel $A\otimes B$? L'auteur démontre une conjecture de Hochschild: dim $(A\otimes B)\leqq$ dim $A+$dim B (voir conjecture 2a et th. 5.1) lorsque A et B ont un élément unité. La question de savoir si

$$(1)\qquad \dim(A\otimes B)=\dim A+\dim B$$

a toujours lieu pour des algèbres avec élément unité est laissée ouverte. Cependant (1) est prouvé quand dim $B=0$, A avec élément unité; et quand dim $B=1$, A et B avec élément unité. Conséquence: soit H une algèbre avec élément unité, de dimension 1 (une telle H existe d'après Hochschild); soit $H^n=H\otimes\cdots\otimes H$ (n fois); alors dim $H^n=n$. Ce résultat, déjà prouvé par Hochschild pour $n=2$, montre qu'il existe des algèbres de toutes dimensions finies. Enfin l'auteur annonce que (1) peut être faux pour des algèbres sans élément unité, et renvoie pour cela à un article ultérieur. *H. Cartan* (Paris).

Krull, Wolfgang. Über geschlossene Bewertungssysteme. J. Reine Angew. Math. **190**, 75–92 (1952).

It is proved in this note that sets of discrete rank-one valuations can be found in algebraic function fields of more than one variable, whose properties give rise to a divisibility theory paralleling portions of the arithmetic in fields of Kroneckerian dimension one, for example. The properties in question, like finiteness and local independence, are modeled after those used by Artin and Whaples, and others. The author's results are couched in the language of valuation theory, but are, as indicated in the note, an elaboration of simple facts in the algebraic geometry of a polynomial domain whose variety is "completed" by introducing a hyperplane at infinity. *O. F. G. Schilling.*

Tornheim, Leonard. Normed fields over the real and complex fields. Michigan Math. J. **1**, 61–68 (1952).

The author gives a new proof, which uses neither complex variable theory nor the completion of F, for the fundamental statement of normed ring theory: A normed field F over the complex field C is C itself. For, if $y \in F$, y non-$\in C$, then $\|1/(y-c)\|$ is a continuous positive function of the complex variable c which is small for large c and therefore takes on its maximum $M>0$ on a closed bounded set $C_0\subset C$. But this is impossible because C_0 contains with any element c_0 a whole circle of radius $r<1/M$ about c_0. To see this, substitute $y-c_0$ for x in the identity

$$\frac{1}{n}\sum_{\nu=1}^{n}\frac{1}{x-e^{2\pi i\nu/n}}=\frac{x^{n-1}}{x^n-r^n}=\frac{1}{x-r(r/x)^{n-1}}$$

and let n tend to infinity. For fixed $r<1/M$ the right side approaches $1/(y-c_0)$. Hence, taking norms and looking at the left side we see that the average value of $\|1/(y-c)\|$ at n equally spaced points on a circle of radius r about c_0 tends to $\|1/(y-c_0)\|=M$. By continuity it follows that $\|1/(y-c)\|=M$ on the whole circle. The same technique is used to prove that a normed field over the reals is either the real or complex numbers. *J. Tate* (Princeton, N. J.).

Theory of Groups

Popova, Hélène. Sur les quasi-groupes dont les logarithmétiques sont groupes. C. R. Acad. Sci. Paris **234**, 2582–2583 (1952).

The logarithmetic L_Q of a non-associative algebra Q consists of quasi-integers [cf. Popova, same C. R. **234**, 1936–1937 (1952); these Rev. **13**, 906] which, although subject to non-associative addition, have associative multiplication. Hence with respect to multiplication L_Q is a group if it is a quasigroup. This is the case when Q is a finite quasigroup if and only if every power permutes the elements of Q. Then the order N of L_Q is not greater than the order n of Q; and if further Q has no proper homomorph or proper subquasigroup, $N=n$. *I. M. H. Etherington* (Edinburgh).

Dyubyuk, P. E. On the number of subgroups of certain categories of finite p-groups. Mat. Sbornik N.S. **30**(72), 575–580 (1952). (Russian)

Let the rank of a p-group be the minimum number of its generators. Then for odd primes p, consider a group P of order p^n and rank >3; and mark the subgroups thereof of rank 3 and of order p^m where $m\geqq 4$. Then if the number of subgroups of order p^k $(1<k<m-1)$ of each such marked subgroup of order p^m is congruent to $1+p+2p^2$, mod p^3, then the number of subgroups of order p^j $(1<j<n-1)$ of P is likewise so congruent. Now let P be a p-group $(p>3)$ of rank >2. Let each subgroup Q of P of rank 3 and order $\geqq p^4$ be such that (1) the number of maximal subgroups of Q of rank 2 is a multiple of p and (2) the number of subgroups of Q of order p^k $(0\leqq k\leqq n-1)$ which are included in any such maximal subgroup of Q of rank 2 is an invariant of Q. Then if $1<m<n-1$, the number of subgroups of P of order p^m is congruent to $1+p+2p^2$, mod p^3. The author shows that this result is a generalization of Theorem 12 of his earlier paper [Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 351–378 (1948); these Rev. **10**, 98]. If P is of order p^n $(p>3)$ of rank 2, no subgroup of which is of rank >2 and if P does not have a cyclic maximal subgroup, then the above result will be valid if the expression $1+p+2p^2$ be replaced by $1+p+p^2$. For related material see Loo-Keng Hua [Sci. Rep. Nat. Tsing Hua Univ. **4**, 313–327 (1947); these Rev. **10**, 8]. *F. Haimo* (St. Louis, Mo.).

Čunihin, S. A. On Π-properties of finite groups. Amer. Math. Soc. Translation no. 72, 32 pp. (1952).

Translated from Mat. Sbornik N.S. **25**(67), 321–346 (1949); these Rev. **11**, 495. In line 23, p. 4, read "count $\{E\}=\mathfrak{E}$ as p-Sylow".

Kuratowski, C., et Mostowski, A. Sur un problème de la théorie des groupes et son rapport à la topologie. Colloquium Math. **2** (1951), 212–215 (1952).

Let A be a Boolean ring (operations $+$ and $\cdot$) and G an abelian group (operation o, neutral element z). The authors raise the general question of determining the structure of the group $\mathfrak{n}_G(A)$ of all functions $\nu(a)$, defined on A with values in G, which satisfy the following conditions: $\nu(1)=z$, $\nu(a+b)=\nu(a)\circ\nu(b)$ if $a\cdot b=0$. They consider particular cases of this problem, especially the case where G is the additive group of integers and A is the ring of all open and closed subsets of certain types of spaces; when the space is a closed (resp., an open) subset of the sphere S_2 of a complex variable, the group $\mathfrak{n}$ (resp. $\mathfrak{n}^*$, obtained when replacing in the definition of $\mathfrak{n}$ the finite additivity condition by an infinite one) has a function-theoretical meaning, given previously by Kuratowski [Topologie, t. II, Warszawa-Wrocław, 1950, p. 418; Fund. Math. **33**, 316–367 (1945), p. 348 (the authors point out that in this paper the group $\mathfrak{n}^*$ has been confused with the group $\mathfrak{n}$); these Rev. **8**, 50; **12**, 517]. The following theorem is proved: if A is the ring of all subsets of a set of power $\mathfrak{m}$, and G is the additive group of integers, the power of the group $\mathfrak{n}_G(A)$ is $2^{2^{\mathfrak{m}}}$. *J. L. Tits* (Brussels).

Mostowski, A. Groups connected with Boolean algebras (partial solution of the problem P 92). Colloquium Math. **2** (1951), 216–219 (1952).

The notations being the same as in the paper reviewed above, the author proves that if A is a ring with an ordered basis [see Mostowski and Tarski, Fund. Math. **32**, 69–86 (1939)], and if M is such a basis, then $\mathfrak{n}_G(A)$ is isomorphic with $G^{M'}$, where $M'=M-\{0, 1\}$. *J. L. Tits* (Brussels).

Gericke, H. Äquivalenz des Satzes von Hajos mit einer Vermutung von Minkowski. Arch. Math. **3**, 34–37 (1952). Please see the corresponding note on p. 1277.

Earlier proofs of the equivalence of the Minkowski conjecture on linear forms (M) and Hajós's theorem on abelian groups (H) have employed the intermediate result that every simple lattice covering of n-dimensional space by

hypercubes is columnated. See, for example, the paper of Hajós [Math. Z. 47, 427–467 (1941); these Rev. 3, 302]. The author seeks to give a direct proof, without considering lattices of hypercubes, for the case when the coefficients of the forms are rational. This he does by using the fact that the determinant of the system of defining relations of the group is equal to its order. In both parts of the proof the author appears to make the assumption that if $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_n$ are generators of a finite abelian group having addition as its operation, then $\sum_k y_k \mathfrak{a}_k = \sum_{i,k} x_i q_{ik} \mathfrak{a}_k$ implies that $y_k = \sum_i x_i q_{ik}$. No justification for this conclusion is given. In the cases $n=2$ and 3 (H) is proved without appealing to (M), the proof in the latter case depending on the assumption mentioned above. *R. A. Rankin* (Birmingham).

Szele, Tibor. Ein Analogon der Körpertheorie für abelsche Gruppen. J. Reine Angew. Math. **188**, 167–192 (1950).

Although the avowed purpose of this paper is to construct for abelian groups an analogue of the Steinitz field theory, the results are, for the most part, expressible in terms of the inner properties of the groups involved and formulations in terms of group extensions appear to be secondary. Let $G \supset H$ where capitals refer exclusively to additive abelian groups. $x \in G$ is said to be algebraic over H if some non-zero $nx \in H$, where n is a natural number. If all the non-zero elements of G are algebraic over H, then G is said to be an algebraic extension of H; and a group H is said to be algebraically closed if it has no proper algebraic extension. Algebraic closure for G is equivalent to each of the following separately: (1) G is complete in that $nG=G$ for all natural numbers n [cf. Baer, Duke Math. J. 3, 68–122 (1937)]; (2) G is a direct summand of every containing group [Baer, Bull. Amer. Math. Soc. **46**, 800–806 (1940); these Rev. **2**, 126]; (3) G is the direct sum of copies of the group of rationals R and/or groups of type p^∞. It turns out that every abelian group is a subgroup of such a group; (4) G has no non-trivial homomorphic images which are of finite order; and (5) G coincides with its Frattini subgroup [Zassenhaus, Lehrbuch der Gruppentheorie, Teubner, Leipzig-Berlin, 1937]. To each G is assigned its rank, its maximum number of linearly independent elements, where linear independence for a set of elements $\{a_k\}$, $k=1, 2, \cdots, j$, means that $\sum \{n_k a_k = 0$ for some natural numbers n_k implies that each $n_k a_k = 0$, and the a_k which are not of 0 order are of prime power order. Groups of rank 1 in this sense are just the subgroups of R and the subgroups of groups of type p^∞. Additional results are: If G is torsion-free, then it is the direct sum of a finite number of copies of R if and only if it has no proper subgroups isomorphic to G. A non-trivial abelian group G has at least one cyclic group of prime order or a group of type p^∞ as a homomorphic image. G has the maximal condition on subgroups if and only if G has a finite number of generators. G has the minimal condition for subgroups if and only if G is a torsion group of finite rank (in the sense of rank as defined in this paper), or, equivalently, if and only if G is the direct sum of a finite number of cyclic groups of prime power order and/or groups of type p^∞. *F. Haimo* (St. Louis, Mo.).

Kertész, A., and Szele, T. On abelian groups every multiple of which is a direct summand. Acta Sci. Math. Szeged **14**, 157–166 (1952).

Let G be an additive abelian group. The authors prove that nG is a direct summand of G for each natural number n if and only if $G=A+B$, a direct sum of subgroups A and B, where (a) $nA=A$ for every natural number n; (b) B lies between its torsion subgroup T and the strong direct sum on the primary components of T as summands; (c) the orders of elements of T are all square-free; (d) like (a) with A replaced by B/T. Every endomorphic image of G is a direct summand of G if and only if, in addition to the above conditions, we have the validity of (e) any given prime p cannot be an order for both an element of A and an element of B. If T is the torsion subgroup of an abelian group G such that (c) holds and (a) is valid with A replaced by G/T, then $G=A+B$ where A is a torsion-free subgroup, and (a), (b), and (d) are valid. The pertinent literature includes R. Baer, Ann. of Math. **37**, 766–781 (1936), and the paper reviewed just above. *F. Haimo* (St. Louis, Mo.).

Welter, C. P. The advancing operation in a special abelian group. Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 304–314 (1952).

The author generalizes the mathematical game of R. Sprague [Math. Z. **51**, 82–84 (1947); these Rev. **9**, 330] to the case of a countable number of compartments. He employs an abelian group, every element of which has order 2, with an ordering relation $<$ given by

$$1<a<b<ab<c<ac<bc<abc<d<ad<\cdots;$$

and he considers (a) the operation E which carries each element of the group onto its immediate successor, (b) F, the inverse of E, and (c) an operation S on five variables:

$$S(x, y; z, u, v) = ExF(Fxy)FxzExF(Fxy)FxuExF(Fxy)Fxv.$$

The main result is that in the generalized Sprague game $S(x, y; z, u, v)=1$ replaces the condition of Sprague which states that the winning position is given by $xyzuv=1$. Note that both the group and the ordering relation on it differ from the corresponding objects in Sprague. *F. Haimo.*

Kulikov, L. Ya. Generalized primary groups. I. Trudy Moskov. Mat. Obšč. **1**, 247–326 (1952). (Russian)

A primary group is an abelian group in which every element has order a power of a (fixed) prime. For primary groups there is a highly developed theory, not shared by abelian groups with elements of infinite order. Let K_p denote the ring of rational numbers with denominator prime to p, Z_p the ring of p-adic integers. A primary group admits K_p and Z_p as operators in a natural way. The author holds that modules over K_p and Z_p are the appropriate generalization of primary groups, and he calls them generalized primary groups. (The reviewer remarks parenthetically that modules over a general valuation ring would have been a useful unifying concept.) This paper is the first of two devoted to generalized primary groups; the introduction gives a preview of part II and it appears from this that the high spot will be existence and uniqueness theorems of the Ulm-Zippin type. Since the present paper is largely a thorough survey of basic concepts, it will be summarized briefly.

In §1 the author studies conditions for an abelian group G to be a generalized primary group; a typical theorem asserts that necessary and sufficient conditions are that $mG=G$ for m prime to p, and that all torsion elements have order a power of p. Under suitable conditions it is further shown that subgroups are automatically submodules, and homomorphisms are automatically module homomorphisms. In §2 a generalized primary group G is called complete if $pG=G$. There exists in any generalized primary group a unique largest complete submodule, it is a direct summand, and its structure is fully known. There is an essentially unique embedding of a generalized primary group in a

complete one. In §3 the author gathers facts concerning "servant" submodules H of G, defined by the property $p^iH = H \cap p^iG$ for all integers i. There is also introduced the companion concept of an "isotype" submodule, for which this equality is further assumed for i ranging over the transfinite ordinals. In §4 the author's theory of basic subgroups [Mat. Sbornik N.S. **16(58)**, 129–162 (1945); these Rev. **8**, 252] is extended from primary groups to generalized primary groups. *I. Kaplansky* (Chicago, Ill.).

Kaplansky, Irving. Some results on abelian groups. Proc. Nat. Acad. Sci. U. S. A. **38**, 538–540 (1952).

Brief announcement of some new theorems on infinite Abelian groups; the full proofs will be contained in a monograph which is in the course of preparation. Specimen theorems: Any isomorphism between the rings of endomorphisms of two primary Abelian groups is induced by an isomorphism between the groups themselves. Shiffman's characterisation [Duke Math. J. **6**, 579–597 (1940); these Rev. **2**, 5] of fully invariant subgroups of primary groups without elements of infinite height can be extended to the case when every two elements can be embedded in a countable direct summand, in particular when the group is countable. All characteristic subgroups are in this case fully invariant, the only exception resulting from a generalisation of results by Shoda [Math. Z. **31**, 611–624 (1930)] and Baer [Proc. London Math. Soc. **39**, 481–514 (1935)]. Any complete module over the p-adic integers (or over any complete discrete valuation ring; completion in the sense of the p-power topology) is the completion of a direct sum of cyclic modules. The cardinal number of cyclic summands appropriate to each order determines the structure of the module completely. Any module over the p-adic integers has a direct summand of rank one. It is indecomposable, therefore, if and only if it has rank one. *K. A. Hirsch.*

Mišina, A. P. On the isomorphism of complete direct sums of abelian groups of rank 1 without torsion. Mat. Sbornik N.S. **31(73)**, 118–127 (1952). (Russian)

Let the group G (or G^*) be the complete direct sum of the torsion-free abelian groups R_α, where $\alpha \in \mathfrak{M}$ (or, respectively, R_β^*, where $\beta \in \mathfrak{N}$), of rank one. Let the types of the components R_α of G satisfy the so-called condition (A). If G is isomorphic to G^*, then for each type s, the subgroup of G, which is the complete direct sum of all those R_α whose type is s, is isomorphic to the corresponding subgroup of G^*, which is the complete direct sum of all those R_β^* whose type is s. Condition (A) is too lengthy to be given here in detail. One significant special case may be mentioned: when there are at most countably many different types of the components R_α. *R. A. Good* (College Park, Md.).

Miller, Clair. The second homology group of a group; relations among commutators. Proc. Amer. Math. Soc. **3**, 588–595 (1952).

The second homology group, $H_2(G)$, of a group G, with integer coefficients, is expressed as the group of relations among formal commutators modulo those relations that hold universally. For general G, define $\langle G, G\rangle$ as the free group on all pairs $\langle x, y\rangle$ of elements from G, and $Z(G)$ as the kernel of the natural mapping of $\langle G, G\rangle$ onto the commutator group (G, G). If G is represented as the quotient of a free group F by a subgroup R, the canonical projection induces a mapping of $Z(F)$ into $Z(G)$, and the quotient is readily shown isomorphic to $H_2(G)$ in Hopf's form, $R \cap (F, F)/(F, R)$. In fact, the author proceeds from an invariant definition of the image of $Z(F)$ in $Z(G)$, by a set of four identities; and the major effort is devoted to showing these identities characterize $Z(F)$ as defined above. The main step is the proof that (with obvious notation) $\langle A \vee B, A \vee B\rangle = \langle A, A\rangle\langle A, B\rangle\langle B, B\rangle$ holds modulo the four identities. For abelian G, the author's expression for $H_2(G)$ simplifies directly to $G \otimes G/D$, the tensor square modulo the diagonal. *R. C. Lyndon* (Princeton, N. J.).

MacLane, Saunders. Duality for groups. Bull. Amer. Math. Soc. **56**, 485–516 (1950).

This paper is an expanded version of an earlier note [Proc. Nat. Acad. Sci. U. S. A. **34**, 263–267 (1948); these Rev. **10**, 9]. *R. C. Lyndon* (Princeton, N. J.).

Green, J. A. A duality in abstract algebra. J. London Math. Soc. **27**, 64–73 (1952).

The "dual" of a theorem about groups and homomorphisms is obtained, according to MacLane [Proc. Nat. Acad. Sci. U. S. A. **34**, 263–267 (1948); these Rev. **10**, 9; see also the preceding review] by inverting the direction of each homomorphism, inverting the order of all products of homomorphisms and replacing homomorphisms onto by isomorphisms into. The author applies this notion of duality to universal algebra to define algebras which are dual to free algebras. *R. D. Schafer* (Philadelphia, Pa.).

Hua, L. K., and Reiner, I. Automorphisms of the projective unimodular group. Trans. Amer. Math. Soc. **72**, 467–473 (1952).

Let $\mathfrak{M}_n$ denote the group of $n \times n$ matrices of determinant ± 1, and let $\mathfrak{M}_n^+$, respectively $\mathfrak{M}_n^-$, stand for the subsets of $\mathfrak{M}_n$ with determinant $+1$, respectively -1; let $\mathfrak{P}_{2n}$ be obtained from $\mathfrak{M}_{2n}$ by identifying $+X$ and $-X$, and let, correspondingly, $\mathfrak{P}^+{}_{2n}$ and $\mathfrak{P}^-{}_{2n}$ be obtained from $\mathfrak{M}^+{}_{2n}$ and $\mathfrak{M}^-{}_{2n}$, respectively. Let X' denote the transpose of X, let $I^{(n)}$ be the identity matrix in $\mathfrak{M}_n$ and let $J_1 = (-1) \dotplus I^{(2n-1)}$. Let $\mathfrak{S}_{2n}$ be the commutator subgroup of $\mathfrak{P}_{2n}$ and $\mathfrak{B}_{2n}$ the group of all automorphisms of $\mathfrak{P}_{2n}$. Then the following theorems are proved: (1) $\mathfrak{S}_{2n} \subset \mathfrak{P}^+{}_{2n}$; for $n = 1$, $\mathfrak{S}_{2n}$ is of index 2 in $\mathfrak{P}^+{}_{2n}$, for $n > 1$, $\mathfrak{S}_{2n} = \mathfrak{P}^+{}_{2n}$. (2) In any automorphism of $\mathfrak{P}_{2n}$, $\mathfrak{P}^+{}_{2n}$ goes into itself. (3) Every automorphism of $\mathfrak{P}_2^+$ is of the form $X \in \mathfrak{P}_2^+ \to AXA^{-1}$ for some $A \in \mathfrak{M}_2$; consequently, every automorphism of $\mathfrak{P}_2$ is of the form $X \in \mathfrak{P}_2 \to AXA^{-1}$ $(A \in \mathfrak{M}_2)$. (4) The generators of $\mathfrak{B}_{2n}$ are: (i) the set of all "inner" automorphisms $\pm X \in \mathfrak{P}_{2n} \to \pm AXA^{-1}$ (where $A \in \mathfrak{M}_{2n}$, not necessarily $A \in \mathfrak{P}_{2n}$), and (ii) the automorphism $\pm X \in \mathfrak{P}_{2n} \to \pm X'^{-1}$. (For the case $n = 1$ see Schreier [Abh. Math. Sem. Univ. Hamburg **3**, 167–169 (1924)].) Theorem (1) follows from Theorem 1 of a previous paper of the authors [same Trans. **71**, 331–348 (1951); these Rev. **13**, 328], hereafter quoted as AUT. Theorem (2) is proved in the present paper for $n = 1$; if $n > 1$, it follows as an immediate corollary from Theorem (1). Theorem (3) is proved, using Theorem 2 from AUT. Theorem (4) is proved by induction on n, using the lemma: In any automorphism τ of $\mathfrak{P}_{2n}$, $J_1^\tau = \pm AJ_1A^{-1}$ for some $A \in \mathfrak{M}_{2n}$. The real difficulty lies in the proof of this lemma, as was the case with the corresponding result (Lemma 2) of AUT. *E. Grosswald* (Philadelphia, Pa.).

Dietz, Helmut. Zur Darstellungstheorie der binären projektiven Gruppe über einem Galoisfeld. Math. Nachr. **7**, 219–256 (1952).

(Ordinary absolutely) irreducible representations of a binary projective group $\mathfrak{H}$ over a finite field are determined

in terms of the generators of their representation-modules. The determination starts with decomposing the group ring of the subgroup $\mathfrak{G}$ of all integral transformations $\alpha z+\beta$ into irreducible left-ideals, which in turn induce (in the manner of Frobenius) left-ideals in the group ring of $\mathfrak{H}$. Then the known degrees and characters of the irreducible representations of $\mathfrak{H}$ [Schur, J. Reine Angew. Math. **132**, 85–137 (1907)] are used to see how the last left-ideals are decomposed into irreducible ones, in fact, each into mutually inequivalent ones fortunately, and to construct generators of the irreducible components. *T. Nakayama* (Nagoya).

Murnaghan, F. D. On the Poincaré polynomial of the full linear group. Proc. Nat. Acad. Sci. U. S. A. **38**, 606–608 (1952).

The Poincaré polynomial of the full linear group of dimension n is

$$P_n(z)=\frac{(1+z)^k}{n!}\int_0^1\cdots\int_0^1 \psi(z)\bar{\psi}(z)\Delta\bar{\Delta}d\phi_1\cdots d\phi_N,$$

where $\psi(z)=\prod_{i<k}(\epsilon_i+z\epsilon_k)$, $\epsilon_k=\exp(2\pi i\phi_k)$, $\Delta=\prod_{i<k}(\epsilon_i-\epsilon_k)$. Weyl [The classical groups, Princeton Univ. Press, 1939; these Rev. **1**, 42] remarks that no one has succeeded in integrating this directly. The author meets the challenge. From the theory of residues this integral is equal to $(-1)^{\frac{1}{2}n(n-1)}/n!$ times the coefficient of $(\epsilon_1\cdots\epsilon_n)^{n-1}$ in the product

$$\phi(z\epsilon_1)\phi(z\epsilon_2)\cdots\phi(z\epsilon_n)\Delta^2,$$

where $\phi(t)=\prod(t+\epsilon)$. This product is expressed as a sum of products each of two alternants. The product of two alternants is expressible as a determinant whose elements are the power sums $s(m)$. In picking out the coefficient of $(\epsilon_1\cdots\epsilon_n)^{n-1}$, each $s(m)$ may be neglected for $m>n-1$. This consideration leads to an inductive relation

$$P_n(z)=(z^{2n-1}+1)P_{n-1}(z),$$

from which the well-known result may be deduced. This proof is immensely simpler than the elaborate procedure described by Weyl. Nevertheless, considerable further simplification is possible. *D. E. Littlewood* (Bangor).

Murnaghan, F. D. On the Poincaré polynomials of the classical groups. Proc. Nat. Acad. Sci. U. S. A. **38**, 608–611 (1952).

The author gives another proof of the formula for the Poincaré polynomial of the full linear group [see the preceding review]. He shows that the coefficient of z^{n^2-k} in $P_n(z)$ is equal to the number of times that the expansion $(\{1^{n-1}\}\{1\})\otimes\{1^k\}$ contains the character $\{k^n\}$. He then remarks that this number is independent of n, and uses the fact to obtain the formula by an inductive procedure. Unfortunately, this result is not quite true; e.g., if $n=4$, the coefficient of z^{n^2-7} is 0, but if $n=5$, it is 1. The author uses a similar method to obtain the Poincaré polynomials of the symplectic and orthogonal groups, but again the generalization to n variables does not seem quite rigorous and is open to similar objections. Nevertheless the methods can be adapted to give sound proofs. *D. E. Littlewood.*

Dieudonné, Jean. On the structure of unitary groups. Trans. Amer. Math. Soc. **72**, 367–385 (1952).

Let K be a sfield with an involution $\xi\to\xi^J$. Then K is said to be of the first or second kind according as J does or does not leave invariant every element of the center Z of K. Let E be an n-dimensional ($n\geqq 2$) right vector space over K and $f(x, y)$ a skew-hermitian form in E; i.e., f is linear in y for each $x\in E$ and $f(y, x)=-f(x, y)^J$ for all x, y. The form f is assumed to be non-degenerate and to have index $\nu\geqq 1$ [cf. Dieudonné, Sur les groupes classiques, Hermann, Paris, 1948; these Rev. **9**, 494]. If K has characteristic equal to 2, then it is also assumed that $f(x, x)$ be of the form $\xi+\xi^J$ for each x. If K is of the second kind, then this last assumption is automatically satisfied.

The unitary group $U_n(K, f)$ is defined to be the group of all one-to-one linear transformations u of E onto E such that $f(u(x), u(y))=f(x, y)$ for all x, y. The author has studied previously [loc. cit.] the structure of $U_n(K, f)$ in the two simplest cases, when K is commutative or K is a reflexive sfield. The present paper is concerned with the general case. Recall that a transvection is a linear transformation of the type $x\to x+a\rho(x)$, where ρ is a non-zero linear form on E with $\rho(a)=0$. Unitary transvections exist if and only if $\nu\geqq 1$. The subgroup of $U_n(K, f)$ generated by the transvections is denoted by T_n. The centers of $U_n(K, f)$ and T_n are denoted respectively by Z_n and W_n. It turns out that Z_n consists of linear transformations of the form $x\to x\lambda$ where $\lambda\in Z$ and $\lambda\lambda^J=1$. We summarize below the main results obtained here: (1) If K has more than 25 elements, then $W_n=T_n\cap Z_n$ and T_n/W_n is simple. This factor group is also simple when K has at most 25 elements except when $K=F_4$, $n=2$ and $n=3$, and $K=F_9$, $n=2$ [loc. cit., p. 70]. (2) When K is of the first kind, of finite rank m^2 over Z, of characteristic $\neq 2$, and such that the space S of symmetric elements in K has dimension $m(m+1)/2$ over Z, then $T_n=U_n(K, f)$. When K is of the second kind, then T_n and $U_n(K, f)$ are always different. (3) If K has characteristic $\neq 2$ and $\nu\geqq 2$ (which implies $n\geqq 4$), then T_n is the commutator subgroup of $U_n(K, f)$. This conclusion does not hold for $n=2$, when K is the sfield of generalized quaternions over Z with J as the unique involution which has Z as its set of symmetric elements. In this case T_2 is the unimodular group $SL_2(Z)$ [loc. cit., p. 30]. *C. E. Rickart* (New Haven, Conn.).

Mautner, F. I. Induced representations. Amer. J. Math. **74**, 737–758 (1952).

The classical Frobenius reciprocity theorem relates the process of passing from a representation of a subgroup of a group to the corresponding "induced representation" of the whole group to the process of passing from a representation of the whole group to its restriction to the subgroup. The author presents here a proof of an infinite generalization in which the group can be any separable locally compact unimodular group and the subgroup is compact. The exact statement will be found in the review of the author's preliminary announcement [Proc. Nat. Acad. Sci. U. S. A. **37**, 431–435 (1951); these Rev. **13**, 205]. The proof depends to a large extent on a series of lemmas which constitute a development of the theory of direct integral decompositions.

The author has asked the reviewer to call attention to an omission in the statement of Lemma 2.1. This statement should contain the additional assumption that M is not a factor of type III. Lemma 2.2 is still true as stated. Since Lemma 2.1 is used only in proving Lemma 2.2, the rest of the argument is not affected. *G. W. Mackey.*

Birkhoff, Garrett. Extensions of Lie groups. Math. Z. **53**, 226–235 (1950).

A short new solution of the Riemann-Helmholtz problem in the large is given. The problem is reduced to the algebraic problem of determining all the $(m+n)$-parameter extensions

of an m-parameter Lie group whose linear representations are known. The author states that the method can be and has been successfully applied to other problems.

S. Chern (Chicago, Ill.).

Singer, I. M. Uniformly continuous representations of Lie groups. Ann. of Math. (2) **56**, 242–247 (1952).

It is shown that every real Lie algebra of skew-adjoint bounded operators on a Hilbert space is the direct sum of a compact semi-simple Lie algebra and an abelian Lie algebra. Various immediate consequences of this are mentioned. In particular, if a Lie group G has a locally faithful unitary representation continuous in the uniform operator topology, then G is the direct product of a compact Lie group and a vector group.

H. C. Wang (Princeton, N. J.).

van Est, W. T. Dense imbeddings of Lie groups. II(I,II). Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math. **14**, 255–266, 267–274 (1952).

Let G, H be Lie algebras. An imbedding of G in H is an isomorphism $\phi: G \rightarrow H$. The imbedding is dense if ϕG is dense in H, i.e., if there exists a Lie group $\mathfrak{H}$ generated by H such that the subgroup generated by ϕG is dense in $\mathfrak{H}$. A Lie algebra H is a (CA)-algebra if its adjoint group (i.e., the group generated by its inner derivations) is a closed linear group. The author shows that every Lie algebra G can be imbedded densely in a (CA)-algebra H. There is a smallest H (the (CA)-completion of G), i.e., there is an H such that if G is imbedded densely in H', there exists a dense imbedding of H in H' consistent with the imbedding of G in H, H'. A Lie algebra G is a (CA)-algebra if and only if in any dense imbedding $\phi: G \rightarrow H$, H is the direct sum of ϕG and a central subalgebra of H. The author introduced the concept of (CA)-algebra in an earlier paper [same Proc. **54**, 321–328 (1951); these Rev. **13**, 432].

P. A. Smith.

Gleason, Andrew M. Groups without small subgroups. Ann. of Math. (2) **56**, 193–212 (1952).

The author proves the following theorem, which, combined with a theorem of Montgomery and Zippin [see the following review], gives a complete solution of Hilbert's fifth problem: Let G be a locally compact group of finite topological dimension. Suppose there is a neighborhood of the identity in G which contains no entire non-trivial subgroup of G. Then G is a Lie group. The proof is mainly based upon the following proposition: Suppose G is a connected locally compact group of finite dimension which contains more than one element and does not have small subgroups. Suppose the center of G is totally disconnected. Then G has a non-trivial continuous representation by linear operators on a finite-dimensional vector space.

An outline of the proof of the above proposition is as follows: The author first constructs a one-parameter family of compact sets $\{U(t)\}$ $(t \geqq 0)$ such that $U(s)U(t) = U(s+t)$ and that, for every σ in G, there exists a number $k = k(\sigma)$ satisfying $\sigma U(t) \sigma^{-1} \subset U(kt)$ for all t. The author then builds an arc $\alpha(s)$ $(0 \leqq s \leqq 1)$ in G with the property $\alpha(s)^{-1}\alpha(t) \in U(|s-t|)$, and also a real continuous function $x(\sigma)$ on G such that $x(\sigma) = 0$ outside a compact set, that $x(e) > x(\sigma)$ for $\sigma \neq e$ and that $|x(\sigma\tau) - x(\tau)| \leqq s$ whenever $\sigma \in U(s)$. The group G operates as usual on the space L_2 of real square-integrable functions on G with respect to a Haar measure of G and G can be topologically imbedded in L_2 by identifying σ in G with the function $\sigma \cdot x(\tau) = x(\sigma^{-1}\tau)$. The arc $\alpha(s)$ is then shown to satisfy a Lipschitz condition in L_2 and therefore it is differentiable at some point. This enables us to construct a non-trivial one-parameter subgroup γ_0 in G such that $\gamma_0(s) \in U(|s|)$ for all s. Now let Γ be the family of all one-parameter subgroups γ which have the property that, for some k, $\gamma(s) \in U(k|s|)$ for all s. Γ is closed under the inner automorphisms of G and the one-parameter groups of Γ are all differentiable arcs in L_2. Denoting by Z the set of tangents of these arcs at e, it turns out that Z is a vector space in L_2 and that the correspondence between one-parameter subgroups and their tangents are one-to-one. Finally, it follows from the finite dimensionality of G, that Z has also a finite dimension. The inner automorphisms of G operating on Γ then induces a representation of G in Z and this representation is not trivial, because the center of G is totally disconnected and Γ contains non-trivial γ_0. This proves the proposition.

The above method of proof is very clear and powerful, and the author states that, using the same method, there is definite hope of proving the further conjecture on the structure of locally compact groups: Every connected locally compact group is a projective limit of Lie groups.

K. Iwasawa (Cambridge, Mass.).

Montgomery, Deane, and Zippin, Leo. Small subgroups of finite-dimensional groups. Ann. of Math. (2) **56**, 213–241 (1952).

The authors prove the following theorem, which, combined with a theorem of Gleason [see the review above], gives a complete solution of Hilbert's fifth problem: Let G be a connected, locally connected, finite-dimensional, locally compact and separable group, and let every proper subgroup of G be a generalized Lie group. Then G contains an invariant generalized Lie group H such that G/H satisfies the same conditions as G and that G/H contains no small subgroups. The essential part of the proof of the theorem is to prove that a group G which satisfies the conditions in the theorem and in addition has only the identity in the center, contains no small subgroups.

The proof of the above proposition is divided into two steps: The authors show first that G does not have infinite compact zero-dimensional subgroups which commute element-wise with some subgroup of positive dimension. From this it follows that all proper subgroups of G of positive dimension are actually Lie groups and also that a compact zero-dimensional subgroup can contain no element of infinite order. Using these results, it is then proved that G contains no small finite groups.

For the proof of the first part, the authors assume that the statement to be proved is false and construct a local Lie group R in G such that R coincides with the identity-component of its normalizer and that R and its conjugates either coincide or have only the identity in common. It is then proved that the set of all conjugates of R sweeps out an open subset of G and that R generates a connected Lie group K in G such that the factor group G_K/K of the normalizer G_K of K by K is an infinite non-discrete zero-dimensional group. The first fact implies that the factor space G/G_K is locally shrinkable and the second one implies that G/G_K contains small closed paths which cannot be locally shrinkable. The contradiction completes the proof of the first part of the above proposition. In the proof of the second part, the authors again assume that the statement is false and construct a local Lie group R, similar to the one above. It then follows that G is locally shrinkable and, consequently, that G contains no small finite groups [cf. Montgomery, Ann. of Math. **52**, 261–271 (1950); these Rev. **13**, 319].

An outline of the results of the present paper was published in Proc. Nat. Acad. Sci. U. S. A. **38**, 440–442 (1952); these Rev. **13**, 821. *K. Iwasawa* (Cambridge, Mass.).

Serre, Jean-Pierre. Le cinquième problème de Hilbert. Etat de la question en 1951. Bull. Soc. Math. France **80**, 1–10 (1952).

This is an exposition of recent results on the fifth problem of Hilbert up to 1951. The problem was solved by Gleason, Montgomery and Zippin in 1952 [see the two preceding reviews]. *K. Iwasawa* (Cambridge, Mass.).

Ganea, Tudor. Covering spaces of homogeneous spaces. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 425–432 (1950). (Romanian. Russian and French summaries)

Let $\bar{H}$ be a closed subgroup of a simply connected group $\bar{G}$ and let h be the natural mapping of $\bar{G}$ onto the coset space $E=\bar{G}/\bar{H}$. Let $(\bar{E}, f)$ be any covering of E. Then there is an open and closed subgroup $\bar{H}_0$ of $\bar{H}$ such that the coset space $\bar{G}/\bar{H}_0$ may be identified with $\bar{E}$. The natural mapping $\bar{h}$: $\bar{G}\to\bar{E}$ then covers h. The different ways of so lifting h correspond to the different subgroups conjugate to $\bar{H}_0$ in $\bar{H}$. $(\bar{E}, f)$ is a regular covering of E if and only if $\bar{H}_0$ is normal in $\bar{H}$. The covering group of $(\bar{E}, f)$ is then $\bar{H}/\bar{H}_0$.
R. H. Fox (Princeton, N. J.).

Ganea, Tudor. The fundamental group of covering spaces. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 433–439 (1950). (Romanian. Russian and French summaries)

Let G be a topological group possessing a universal covering group $\bar{G}$ and let H be a closed locally connected subgroup of G, so that the coset space $E=G/H$ has a universal covering space $\bar{E}$. Let H_0 be the component of the identity in H. The author studies the relation between H/H_0 and $\pi(G/H)$. This had been done previously by Pontrjagin [Topological groups, Princeton Univ. Press, 1939, p. 232, example 61; these Rev. **1**, 44] for the case H discrete and G simply connected. *R. H. Fox* (Princeton, N. J.).

Ganea, Tudor. Zur Multikohärenz topologischer Gruppen. Math. Nachr. **7**, 323–334 (1952).

The degree of multicoherence $r(E)$ of a space E is understood in the sense of A. H. Stone [Trans. Amer. Math. Soc. **65**, 427–447; **66**, 389–406 (1949); these Rev. **11**, 44, 45]. A covering space $(\tilde{E}, f)$ of E is called binary if $E=E_1\cup E_2$ where E_i is an open and connected subset of E and is evenly covered by $(\tilde{E}, f)$. Theorem 4. A connected, locally connected space X is unicoherent (i.e., $r(X)=0$) if and only if any map φ of X into any connected, locally connected space Y can be lifted to a map $\tilde{\varphi}$ of X into any given binary covering $(\tilde{Y}, g)$ of Y in such a way that, for a given point $x_0\in X$, $\tilde{\varphi}(x_0)$ is a prescribed point of $g^{-1}\varphi(x_0)$. Theorem 3. If E is connected and locally connected and $r(E)\geqq n$, then E has a binary regular covering space $\tilde{E}$ whose covering group is any prescribed group of rank n. Theorem 2 (consequence of Theorems 3 and 4). Let $\mathfrak{H}$ be a group of autohomeomorphisms of a connected, locally connected unicoherent space X and let Y be the decomposition space. If $r(Y)\geqq n$, then $\mathfrak{H}$ has a retract isomorphic to the free group of rank n. Theorem 1 (consequence of Theorem 2). If a topological group G is a homomorph of a connected, locally connected, unicoherent topological group, then $r(G)\leqq 1$. The author remarks, in connection with the principal result, Theorem 1, that Eilenberg [Fund. Math. **27**, 153–190 (1936)] has proved $r(G)\leqq 1$ for any topological group G that is a Peano space, and that there is a connected, locally connected (but not compact) group G, discovered by Jones [Bull. Amer. Math. Soc. **48**, 115–120 (1942); these Rev. **3**, 229] for which $r(G)=\infty$. *R. H. Fox* (Princeton, N. J.).

NUMBER THEORY

Coxeter, H. S. M. Rouse Ball's unpublished notes on three fours. Scripta Math. **18**, 85–86 (1952).

A recreational item in which every integer $\leqq 90$ is written in terms of three fours using the symbols for factorial and subfactorial. *D. H. Lehmer* (Los Angeles, Calif.).

Moessner, Alfred. On the multiple identity

$$x_1^n+x_2^n+x_3^n+x_4^n+x_5^n=y_1^n+y_2^n+y_3^n+y_4^n+y_5^n$$

for $n=1, 3, 5, 7$. Scripta Math. **18**, 90–91 (1952).

Moessner, Alfred. Alcuni problemi diofantei elementari. Boll. Un. Mat. Ital. (3) **7**, 185–187 (1952).

Gloden, A. Notes on Diophantine equations. Scripta Math. **18**, 87–89 (1952).

Comments are made on various Diophantine equations, such as how to obtain an infinity of solutions from one solution of $A(x^2+y^2+z^2)=B(xy+xz+yz)$. *I. Niven.*

Skolem, Th. A simple proof of the condition of solvability of the Diophantine equation

$$ax^2+by^2+cz^2=0.$$

Norske Vid. Selsk. Forh., Trondheim **24** (1951), 102–107 (1952).

The theorem proved is the well-known result of Legendre that if abc is square-free, necessary and sufficient conditions for the non-trivial solvability of (*) $ax^2+by^2+cz^2=0$ are that a, b, c are not of like sign and that $-bc$, $-ca$, $-ab$ are quadratic residues modulo a, b, c respectively. This latter condition is proved equivalent to the reducibility of $ax^2+by^2+cz^2$ (mod abc) under the hypothesis that abc is square-free, and this enables the author to treat (*) by discussing bounds for the least nontrivial solution of a linear congruence. Thus a lemma is introduced which is a special case of a theorem on linear congruences of A. Brauer and R. L. Reynolds [Canadian J. Math. **3**, 367–374 (1951); these Rev. **14**, 21]. As a corollary the treatment yields bounds for the minimum solution of (*) which, however, are not quite as good as those of L. Holzer [ibid. **2**, 238–244 (1950); these Rev. **12**, 11]. Another simple proof of Legendre's theorem was given recently by L. J. Mordell [Monatsh. Math. **55**, 323–327 (1951); these Rev. **13**, 534], who also employs bounds for the solution of a congruence, a quadratic congruence however. *I. Niven.*

Ljunggren, Wilhelm. On the Diophantine equation

$$x^2+4=Ay^4.$$

Norske Vid. Selsk. Forh., Trondheim **24** (1951), 82–84 (1952).

It is proved that the equation $x^2+4=Ay^4$ has at most one solution in positive integers x and y with x odd. The proof employs properties of units in the quadratic field $R(\sqrt{A})$.
I. Niven (Eugene, Ore.).

Bini, Umberto. La risoluzione delle equazioni $x^n \pm y^n = M$ e l'ultimo teorema di Fermat. Archimede 4, 50–57 (1952).

The author gives a number of equations of the form $x^n+y^n=M$ which are impossible in positive coprime integers x, y. The arguments are based on inequalities for $x+y$ and on certain divisibility properties of x^n+y^n.

D. H. Lehmer (Los Angeles, Calif.).

Aubert, K. E. Functions which represent prime numbers. Norsk Mat. Tidsskr. 34, 42–44 (1952). (Norwegian)

Expository article. *I. Niven* (Eugene, Ore.).

Palamà, Giuseppe. Numeri primi e composti contenuti nella forma $1848x^2+y^2$ dell'intervallo 11 000 000–11 100 000. Boll. Un. Mat. Ital. (3) 7, 168–171 (1952).

The author lists the 202 prime, and the 121 composite, numbers of the idoneal form $1848x^2+y^2$ among the first 100000 numbers of the 12th million. In the case of the primes, the values of x and y are also given.

D. H. Lehmer (Los Angeles, Calif.).

Wright, E. M. The elementary proof of the prime number theorem. Proc. Roy. Soc. Edinburgh. Sect. A. 63, 257–267 (1952).

A complete elementary proof of the prime number theorem is given (following the method of A. Selberg) in which no previous knowledge of number theory is presumed. Selberg's fundamental formula

$$(1) \qquad \psi(x)\log x+\sum_{mn\leq x}\Lambda(m)\Lambda(n)=2x\log x+O(x)$$

is proved by a procedure which is essentially the same as the proof given by Selberg in that it stems from a consideration of the sum

$$(2) \qquad \sum_{d|n}\mu(d)\log^2\frac{x}{d}.$$

However the proof is arranged in such a manner as to avoid explicit reference to any sums involving the divisor function, and is in this direction very similar to a proof given by the reviewer [Ann. of Math (2) 51, 485–497 (1950); these Rev. 11, 419]. In this connection it should be pointed out that the use of (2) is slightly misleading, and that the arithmetic identity which underlies Selberg's formula is

$$(3) \qquad \sum_{d|n}\mu(d)\log^2\frac{n}{d}=\Lambda(n)\log n+\sum_{d|n}\Lambda(d)\Lambda\left(\frac{n}{d}\right).$$

This identity can be proved arithmetically by observing that

$$\log^2 n=\log n\sum_{d|n}\Lambda(d)=\sum_{d|n}\Lambda(d)\log\frac{n}{d}+\sum_{d|n}\Lambda(d)\log d$$
$$=\sum_{d|n}\left\{\Lambda(d)\log d+\sum_{c|d}\Lambda(c)\Lambda\left(\frac{d}{c}\right)\right\}.$$

Applying the Möbius inversion formula to this yields (3).

In his derivation of the prime number theorem from (1) the author follows Selberg's method except that in the concluding argument sums are replaced by integrals. This tends to elucidate somewhat the idea behind Selberg's proof.

H. N. Shapiro (New York, N. Y.).

Turán, P. On the remainder-term of the prime-number formula. II. Acta Math. Acad. Sci. Hungar. 1, 155–166 (1950). (English. Russian summary)

The usual derivations of estimates of $D(x)=\pi(x)-\mathrm{Li}(x)$ are based on the knowledge of zero-free regions for $\zeta(s)$. For example, it is known that if (i) $\zeta(s)\neq 0$ in $\sigma>1-c/\log^\beta|t|$, $|t|>c'$, then (ii) $D(x)=O\{x\exp(-c''(\log x)^{1/(1+\beta)})\}$, provided $0<\beta\leq 1$. Here, the author proves the converse statement that if $0<\beta<1$, then (ii) implies (i). Consequently, it is not possible to improve (ii) (except possibly for the constant c'') unless (i) is improved.

The proof depends on obtaining upper and lower bounds for

$$S=\sum_{n\geq\xi}\Lambda(n)n^{-s}\log^{k+1}n/\xi.$$

Beginning with $\sum\Lambda(n)n^{-it}$, the summation being over a subinterval of $(N, 2N)$, the author replaces $\Lambda(n)$ by $\psi(n)-\psi(n-1)$, uses partial summation, and applies hypothesis (ii) in the form that $\psi(x)-x$ has the same estimate as that given for $D(x)$. By using partial summation, $\sum\Lambda(n)n^{-s}$ is now estimated. Now the function $f_2(s,\eta)=\sum_{n>\eta}\Lambda(n)n^{-s}$ is estimated from above by breaking the range $n>\eta$ up into a number of sub-intervals of the form $(\eta 2^j, \eta 2^{j+1})$. Since $S/(k+1)=\int_\xi^\infty f_2(s,\eta)\eta^{-1}\log^k(\eta/\xi)d\eta$, an upper bound for S is thereby obtained. S is now expressed as the sum of three terms one of which is $S_1=-(k+1)!\sum_\rho \xi^{\rho-s}/(s-\rho)^{k+2}$. Under the assumption that there is a zero $\rho^*=\sigma^*+it^*$ for which $\sigma^*>1-d/\log^\beta t^*$, $t^*>d'$, the author readily gives upper bounds for the other two terms. There is also no trouble about estimating the contribution to S_1 of all those terms which fail to satisfy both the conditions: (iii) $|t_\rho-t^*|<6(\sigma_1-\sigma^*)$ and $\sigma_\rho\geq 1-3(\sigma_1-\sigma^*)$, where $\sigma_1=1+10d/\log^\beta t^*$. Using all these upper bounds, the author concludes that

$$\left|\sum_{(iii)}e^{(\rho-\rho^*)/(\sigma_1-1)}\{(s-\rho^*)/(s-\rho)\}^{k+2}\right|<d''e^{k+2}(t^*)^{-7/8}$$

where k is merely restricted by the condition

$$\log t^*\leq k+2\leq(5/4)\log t^*.$$

At this point the author applies a previous result of his [same Acta 2, 39–73 (1951); these Rev. 13, 742] which gives a lower bound for sums of the form $|s_1^{k+2}+\cdots+s_n^{k+2}|$. This result enables the author to choose k in such a fashion that he now finds that $\sigma^*<1-4d/\log^\beta t^*$; this contradicts the earlier assumption that $\sigma^*>1-d/\log^\beta t^*$ Trud. Mat. Steklov

The author remarks that routine calculations based on Vinogradov's book [~~Foundations of the theory of numbers, 5th ed., Moscow-Leningrad, 1949~~; these Rev. 10, 599] establish that $\zeta(s)\neq 0$ when $\sigma>1-f/(\log|t|\log\log|t|)^{2/3}$. Actually, the best published results [T. Flett, Quart. J. Math., Oxford Ser. (2) 2, 26–52 (1951); these Rev. 13, 209; E. Titchmarsh, The theory of the Riemann zeta function, Oxford, 1951; these Rev. 13, 741] are based on Vinogradov's work and have the exponent 2/3 replaced by 3/4. The reviewer is inclined to doubt that 2/3 can be obtained by routine calculation. *L. Schoenfeld* (Urbana, Ill.).

dos Reis, Manuel. On conjectured asymptotic formulas concerning the distribution of prime numbers. Gaz. Mat., Lisboa 12, no. 50, 83–90 (1951). (Portuguese)

Using the sieve method in conjunction with some probabilistic arguments, several conjectural formulas are established. Some of them correspond to well-known theorems (the prime number theorem for all integers and for arithmetic progressions; the Hardy-Littlewood-Vinogradov asymptotic formula for the number of representations of an odd number as a sum of three odd primes [Vinogradov, Mat. Sbornik 2(44), 179–195 (1937)]), others were conjectured by Hardy and Littlewood [Acta Math. 44, 1–70 (1923)] and others

still, of the same general type, seem new, as, e.g.,

$$P(n)\sim n^{1/4}(\log n)^{-1}\prod_{p\geqq 3}\{1-(-1/p)(p-1)^{-1}\}\times\prod_{p\equiv 1\ (\mathrm{mod}\ 4)}\{1-(-1/p)_4\cdot 2/(p-2)\}$$

for the number of primes not exceeding n and of the form x^4+1. Here $(-1/p)_4=1$ (respectively, $=-1$) if -1 is biquadratic residue (resp., non-residue) mod p. The fact that some of the formulas obtained by his method are known to be true indicates, in the author's opinion, that also the others have a high degree of probability of being true. Some of the arguments used in the paper do not seem very convincing to the reviewer, even when considered as purely heuristical ones. *E. Grosswald* (Philadelphia, Pa.).

Rai, T. On a problem of additive theory of numbers. II. Math. Student 19 (1951), 113–116 (1952).

Methods and aims are analogous to those of a previous paper of the same author [Math. Student 15, 25–28 (1948); these Rev. 10, 431]. Let $N(k)$ be the least value of n such that (1) $\sum_{i=1}^{m}x_i^k=\sum_{i=1}^{n}y_i^k$ has a solution with $m=n$. Let $\delta(k)$, resp. $\theta(k)$, be the least value of s such that there exists a constant $c=c(k)\neq 0$ and $c=\sum_{i=1}^{s}\epsilon_i x_i^k$, $\epsilon_i=\pm 1$, has infinitely many integer, resp. rational, solutions. Let $\gamma(k)$ be the least value of n such that (1) has infinitely many solutions with $m<n$. From numerical identities the author derives inequalities such as $N(18)\leqq 54$, $\beta(17)\leqq 29$, $\gamma(23)\leqq 83$, $\delta(9)\leqq 15$, $\theta(13)\leqq 21$. *N. G. W. H. Beeger* (Amsterdam).

Gupta, Hansraj. A note on sums of powers. Math. Student 19 (1951), 117 (1952).

Addendum to a paper with the same title and by the same author [Proc. Indian Acad. Sci., Sect. A. 4, 571–574 (1936)]. $M(k)$ denotes the least value of n for which $\sum_{i=1}^{n}a_i^m=\sum_{i=1}^{n}b_i^m$, $m=1, 2, 3, \cdots, k(\neq k+1)$ has a solution. By means of two numerical identities he proves $M(21)\leqq 80$. Proceeding as in Rai's paper [see the preceding review] it can be shown that $\gamma(23)\leqq 81$.

N. G. W. H. Beeger (Amsterdam).

Hovanskiĭ, A. N. Some identities connected with Bernoulli numbers. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk 1, 93–94 (1948). (Russian)

The umbral calculus of Lucas is used to derive the general identity

$$f(x+t(k+B))-f(x+tB)=t\sum_{m=0}^{k-1}f'(x+mt)$$

where $f(u)=a_0+a_1u+a_2u^2+\cdots$ is a formal power series and B the umbra of the Bernoulli numbers $B^0=1$, $B^1=-\frac{1}{2}$, $B^2=\frac{1}{6}$, $B^3=0$, etc. The identity is a linear transformation of a result of Lucas [Théorie des nombres, Gauthier-Villars, Paris, 1891]. Several examples of special cases are given.

D. H. Lehmer (Los Angeles, Calif.).

Carlitz, L. Some theorems on Bernoulli numbers of higher order. Pacific J. Math. 2, 127–139 (1952).

The Bernoulli numbers of order k are defined by

$$\left(\frac{t}{e^t-1}\right)^k=\sum_{m=0}^{\infty}\frac{t^m}{m!}B_m^{(k)}\quad(|t|<2\pi).$$

S. Wachs [Bull. Sci. Math. 71, 219–232 (1947); these Rev. 10, 101] has shown that $B_{p+3}^{(p+1)}\equiv 0 \pmod{p^2}$ for p (prime) $\geqq 3$. Here it is proved that $B_{p+2}^{(p+1)}\equiv 0 \pmod{p^3}$ for $p>3$, and that $B_p^{(p)}\equiv\frac{1}{2}p^3 \pmod{p^3}$ for $p\geqq 3$. Other results connect $B_m^{(p)}$ with the ordinary Bernoulli numbers by congruences mod various powers of p. Except for one special form of m (mod $p(p-1)$), $B_m^{(p)}$ is integral (mod p), $p\geqq 3$.

N. J. Fine (Philadelphia, Pa.).

Slater, L. J. Further identities of the Rogers-Ramanujan type. Proc. London Math. Soc. (2) 54, 147–167 (1952).

The author has recently outlined a method for obtaining identities of the Rogers-Ramanujan type [same Proc. 53, 460–475 (1951); these Rev. 13, 227]. Here he presents a list of 130 such results, some of which have previously been obtained by Rogers, Jackson, and Bailey. *N. J. Fine.*

Bailey, W. N. A further note on two of Ramanujan's formulae. Quart. J. Math., Oxford Ser. (2) 3, 158–160 (1952).

In a recent note [same Quart. 3, 29–31 (1952); these Rev. 13, 725] the author has supplied a missing link in an alternate proof by Ramanujan of the identity

$$\sum_{n=0}^{\infty}p(5n+4)x^n=5\prod_{n=1}^{\infty}\frac{(1-x^{5n})^5}{(1-x^n)^6}.$$

In that note he used the known sum of a well-poised bilateral basic series ${}_6\Psi_6$. Here he obtains the same result by using a well-known formula in elliptic functions,

$$\wp(v)-\wp(u)=\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}.$$

[The reviewer remarks that a similar approach has led him to a proof of Ramanujan's congruence $p(7n+5)\equiv 0 \pmod 7$; the corresponding identity differs from Ramanujan's.]

N. J. Fine (Philadelphia, Pa.).

Gustin, W. An operatorial characterization of certain partition polynomials. Proc. Amer. Math. Soc. 3, 31–35 (1952).

Let I_n ($n\geqq 1$) be the set of integers r such that $0\leqq r<n$. A sequence of x non-empty sets $(N_1, \cdots, N_x)$ of non-negative integers is an x-partition of I_n if every r in I_n can be uniquely expressed in the form $r=r_1+\cdots+r_x$, each r_k belonging to N_k, and if every r of this form is in I_n. Let $f_n(x)$ be the number of x-partitions of I_n. In a paper not yet published the author has shown that

$$(*)\qquad xf_n(x)=1+(x-1)\sum_{d|n}f_d(x),$$

so that $f_n(x)$ is determined uniquely as a polynomial in x with integral coefficients. The author gives a combinatorial solution of (*) in terms of the generalized Möbius functions $\mu_k(n)$ generated by $\zeta^{-k}(s)$. The main result of the paper, however, is that $f_n(x)=F_{k_1}\cdots F_{k_m}[1]$, where n has a prime factorization with exponents $k_1, \cdots, k_m$, and F_k denotes the operator $(k!)^{-1}x^kD^k(x-1)^k$. *N. J. Fine.*

Auluck, F. C., and Haselgrove, C. B. On Ingham's Tauberian theorem for partitions. Proc. Cambridge Philos. Soc. 48, 566–570 (1952).

Under conditions not set forth here, Ingham [Ann. of Math. (2) 42, 1075–1090 (1941); these Rev. 3, 166] obtained the asymptotic form of $P_h(u)=[P(u+h)-P(u)]/h$ where $P(u)$ is the number of solutions in integers $n_i\geqq 0$ of the inequality $n_1\lambda_1+n_2\lambda_2+\cdots+n_r\lambda_r<u$. The present paper removes the Ingham condition that h be such that $P_h(u)$ is an increasing function of u, and replaces the Ingham condition $0<\lambda_1<\lambda_2<\lambda_3<\cdots$ by the weaker condition

$0<\lambda_1\leqq\lambda_2\leqq\lambda_3\leqq\cdots$. Riesz means of second and higher orders enable the authors to overcome difficulties produced by weakening the hypotheses. *R. P. Agnew.*

Iseki, Kanesiroo. A proof of a transformation formula in the theory of partitions. J. Math. Soc. Japan **4**, 14–26 (1952).

The partition function $p(n)$ is generated by the infinite product $f(x)=\prod_1^\infty(1-x^n)^{-1}=1+\sum_1^\infty p(n)x^n$, whose logarithm, $F(x)$, satisfies the transformation formula (1): $F(x)=F(\bar{x})+\log z^{1/2}+\pi/(12kz)-\pi z/(12k)+\pi is(h,k)$, where $x=\exp(2\pi ih/k-2\pi z/k)$, $\bar{x}=\exp(2\pi iH/k-2\pi/(kz))$, h,k,H are integers with $(h,k)=1$, $hH\equiv-1 \pmod k$, $\Re(z)>0$, and $s(h,k)=\sum_{m=1}^{k-1}(m/k)(hm/k-[hm/k]-\frac{1}{2})$ is a Dedekind sum. Formula (1) is due to Dedekind and is one of the fundamental results in the theory of elliptic modular functions and in the asymptotic theory of partitions. Rademacher [J. Reine Angew. Math. **167**, 312–336 (1932)] gave a proof of (1), basing it on the Mellin transform and the functional equation of the Hurwitz zeta function.

The author gives a longer but somewhat more elementary proof of (1), using theta functions instead of zeta functions. As in Rademacher's proof, the double series expansion $F(x)=\sum_{m,n}m^{-1}x^{mn}$ is used, but the author keeps z real and positive and treats the real and imaginary parts of (1) separately. The restriction $z>0$ is later removed by analytic continuation. The elementary integral

$$\int_0^\infty \exp(-t^2-a^2/t^2)dt=\tfrac{1}{2}\pi^{1/2}e^{-2a}\quad(a>0)$$

plays essentially the same role in this proof that the Mellin integral for e^{-x} played in Rademacher's proof, making possible a definite integral representation for $\Re F(x)$ in terms of theta functions. The transformation theory of theta functions then yields

$$\Re F(x)=\Re F(\bar{x})+\log z^{1/2}+\pi/(12kz)-\pi z/(12k).$$

The author then shows that $\Im F(x)-\Im F(\bar{x})$ is a constant independent of z, again via theta functions, and then uses Dedekind's method to show that this constant is $\pi s(h,k)$, and the proof is complete.

The latter part of the paper contains a proof of the reciprocity law for Dedekind sums based on contour integration. This proof has already been given by Rademacher [Mat. Fiz. Lapok, **40**, 24–34 (1933)]. *T. M. Apostol.*

Durst, L. K. The apparition problem for equianharmonic divisibility sequences. Proc. Nat. Acad. Sci. U. S. A. **38**, 330–333 (1952).

A sequence of integers (h): $h_0, h_1, \cdots, h_n, \cdots$, satisfying $h_{m+n}h_{m-n}=h_{m+1}h_{m-1}h_n^2-h_{n+1}h_{n-1}h_m^2$ and such that $n|m$ implies $h_n|h_m$ is called an elliptic divisibility sequence [see M. Ward, Amer. J. Math. **70**, 31–74 (1948); these Rev. **9**, 332]. If $l|k$, $m|h_k$, $m\nmid h_l$, then k is the rank of apparition of m in (h) [see Hall, ibid. **58**, 577–584 (1936)]. If (h) is a general elliptic divisibility sequence, it can be parametrized by $h_n=\psi_n(u)=\sigma(nu)/\sigma(u)^{n^2}$, $\sigma(u)$ being Weierstrass' sigma function. If, in particular, the corresponding periods satisfy $\omega_2/\omega_1=\rho=\frac{1}{2}(-1+\sqrt{-3})$, the Weierstrass functions are called equianharmonic and $g_2=0$. Let E be the ring of integers $\mu=a+\rho b$, a, b rational integers, then $\psi_\mu(u)=P_\mu(z,g_3)$ if the norm $N\mu$ is an odd rational integer and

$$\psi_\mu(u)=\wp'(u)P_\mu(z,g_3)$$

if $N\mu$ is even, with $P_\mu(z,g_3)$ a polynomial in z over the polynomial ring $E[g_3]$. Let z and g_3 be fixed rational integers, let $\mathfrak{p}$ be a prime ideal of E and let λ be an integer of E. If $P_\lambda(z,g_3)\equiv0 \pmod{\mathfrak{p}}$, the integer λ is called a zero of $\mathfrak{p}$. A zero of $\mathfrak{p}$ with minimum positive norm is called the rank of apparition of $\mathfrak{p}$. If $M(\delta)$ is the Möbius function for the principal ideal ring E, define $Q_\mu(z)=\prod_{(\delta)|(\mu)}P_{\mu/\delta}(z)^{M(\delta)}$ and let F_μ be the root field of $Q_\mu(z)$. Then the main results of the paper are: (1) If n is an odd rational integer, then the cyclotomic polynomial $C_n(x)$ is reducible in F_n; (2) if $p\equiv2 \pmod 3$ and α is a rank of apparition of p, then $\alpha=2^cb$ or $\alpha=2^cb(1-\rho)$, where $b=1, 3$, or is an odd divisor of (p^e-1), with $c\geqq0$ and $e\leqq\phi(b)$, both rational integers.

E. Grosswald (Philadelphia, Pa.).

Rankin, R. A. The scalar product of modular forms. Proc. London Math. Soc. (3) **2**, 198–217 (1952).

The author develops methods for effectively calculating the scalar product (f,g) of two modular forms of negative dimension $-k$ belonging to the full group and having unity as multiplier system. His formula when f, g are both cusp forms, for example, is a linear combination of the values of certain Dirichlet series at integral points within their half-plane of absolute convergence. The coefficients in the linear combination are obtained by expressing g, say, in terms of a certain basis of the cusp forms of dimension $-k$. The special case in which the basis consists of one function ($k=12, 16, 18, 20, 22, 26$) is discussed separately and leads to certain identities involving the Fourier coefficients of the modular forms, e.g.,

$$\tau(h)h^{-11}+240\sum_{m=1}^\infty\sigma_3(m)\tau(m+h)(m+h)^{-11}=0,\quad h>0.$$

J. Lehner (Los Alamos, N. M.).

Iwasawa, Kenkiti, and Tamagawa, Tsuneo. Correction. On the group of automorphisms of a function field. J. Math. Soc. Japan **4**, 100–101 (1952).

See same J. **3**, 137–147 (1951); these Rev. **13**, 325.

✶Šafarevič, I. R. A new proof of the Kronecker-Weber theorem. Trudy Mat. Inst. Steklov., v. 38, pp. 382–387. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

A short proof is given of the theorem that every abelian extension of the field of rational numbers is contained in a cyclotomic field. The proof is based on an analogous result for p-adic fields. *W. H. Mills* (New Haven, Conn.).

Inaba, Eizi. On the imbedding problem of normal algebraic number fields. Nagoya Math. J. **4**, 55–61 (1952).

Given are a field k, a normal extension field K of finite degree over k with the Galois group G, and an extension $\bar{G}$ of a finite group H by means of G. The imbedding problem deals with the question under what conditions there exist normal fields L over k such that the corresponding Galois group is isomorphic to $\bar{G}$ with the subfield K corresponding to the subgroup H of $\bar{G}$. If H is soluble and if H_1 is the group preceding $\{1\}$ in a principal series of $\bar{G}$ through H, then H_1 is abelian of type $(p, p, \cdots, p)$ for some prime p. For this reason, the author assumes that H itself is abelian of type $(p, p, \cdots, p)$. The more general case of a soluble H can be reduced to a repeated application of the special case. The extension $\bar{G}$ of H by means of G is determined by a representation $\Lambda(g)$ of G by automorphisms of H and by a factor set $A(s,t)$, s, $t\in G$, with values in H. In our case, $\Lambda(g)$ can be interpreted as a linear representation of G in the Galois

field Γ with p elements and every value of $A(s, t)$ as a vector with coefficients in Γ. If $\Lambda(g)$ is irreducible, that is, if H is a minimal normal subgroup $\neq \{1\}$ of $\bar{G}$, then $\bar{G}$ is said to be irreducible. As already remarked, it would be sufficient to consider only this case. On the other hand, if $\Lambda(g)$ is the representation of G induced by the 1-representation of a p-Sylow subgroup S of G, then $\bar{G}$ is said to be a regular extension. From now on, it is assumed that S is a normal subgroup of G. Then a regular extension is determined by a factor set $a(\sigma, \tau)$, $\sigma, \tau \in S$, with values in Γ;

$$a(\sigma, \tau)+a(\sigma\tau, \varphi)=a(\sigma, \tau\varphi)+a(\tau, \varphi).$$

The author assumes further that k is an algebraic number field of finite degree and that a pth root of unity ζ lies in k; the first of these assumptions is not used very strongly. The elements $\zeta^{a(\sigma,\tau)}$ of k form a multiplicative factor set. A necessary and sufficient condition for the existence of a solution of the regular extension problem under the given assumptions is that this factor set split in K. To every irreducible extension problem, there belongs a regular extension problem such that the solubility of the latter problem implies that of the former problem. *R. Brauer.*

Iseki, Kanesiroo. Über die negativen Fundamentaldiskriminanten mit der Klassenzahl Zwei. Nat. Sci. Rep. Ochanomizu Univ. 3, 23–29 (1952).

If an imaginary quadratic field has class number 2 and discriminant $-\Delta$, then Δ is of one of the forms pq, $4p$, and $8p$, where p and q are distinct odd primes. There are exactly 18 such values of $\Delta<6000$, namely 15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232, 235, 267, 403, and 427. *W. H. Mills* (New Haven, Conn.).

Whaples, G. Generalized local class field theory. I. Reciprocity law. Duke Math. J. 19, 505–517 (1952).

Using approximation methods the author proves that if a field F is complete under a discrete valuation with algebraically closed residue class field, then every element of F is norm from any finite extension. Applying this result to the completion $\bar{W}$ of the infinite unramified extension W of a field k satisfying the axioms for local class field theory, and descending to W by a continuity argument, he shows that every cyclic 2-cocycle class c over k has an unramified splitting field. Corresponding to a selected pseudo-Frobenius substitution generating the Galois group of W over k there is an isomorphism $c\rightarrow\rho(c)$ of the group of cyclic 2-cocycle classes onto the rationals (mod 1). For an abelian extension K/k the reciprocity law map $\alpha\rightarrow(\alpha, K/k)$ of k^* onto the Galois group G is defined by duality as the unique homomorphism for which $\chi((\alpha, K/k))=\rho(\alpha\cup\delta\chi)$ for each character χ of G. The translation law then amounts to a special formula for the transfer of cyclic 2-cocycles. *J. Tate.*

Hasse, Helmut. Rein-arithmetischer Beweis des Siegelschen Endlichkeitssatzes für binäre diophantische Gleichungen im Spezialfall des Geschlechts 1. Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. 1951, no. 2, 19 pp. (1952).

Suppose that $f(x, y)=0$ is an irreducible equation with coefficients in an algebraic number field Ω, which defines an elliptic field K. The author proves, using his arithmetic theory of elliptic function fields, the elliptic case of Siegel's theorem that there exists only finitely many pairs of integers $a, b \in \Omega$ for which $f(a, b)=0$. It is the aim of this paper to reformulate clearly the essential component steps of the proof for the elliptic case so as to obtain hints for further algebraization of Siegel's original proof which referred strongly to the analytically founded theory of abelian functions. Thus the author first works out by means of Weil's theory of distributions a birationally invariant formulation of an intermediate problem as Hypothesis (A_n): There exists an integral square-free divisor $\mathfrak{z}_n$ of K/Ω of degree n^2 such that $\mathfrak{z}_n(\mathfrak{p})\doteq 1$ (equivalence meant in the sense of Weil) for infinitely many prime divisors $\mathfrak{p}$ of K. It is shown that Weil's finiteness theorem in a weakened form (if D is the divisor class group of degree 0, then D/D^n is finite) implies the validity of (A_n) as a consequence of (A_1), where (A_1) is a birationally invariant form of the hypothesis which is customarily shown to lead to a contradiction. Thus a recasting of Siegel's method from the theory of diophantine approximations is employed to set (A_n), for sufficiently large n, into contradiction to the theorem of Thue-Siegel. As to the generalization for arbitrary genus it will be useful to add to the author's bibliography the following papers which deal precisely with the necessary extensions of the elliptic case so that a complete algebraization of some essential steps in Weil's thesis [Acta Math. 52, 281–315 (1929)] become evident: A. Weil, Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948; Ann. of Math. 53, 412–444 (1951); these Rev. 10, 621; 13, 66; W.-L. Chow, Amer. J. Math. 72, 247–283 (1950); these Rev. 11, 615. *O. F. G. Schilling* (Chicago, Ill.).

Hasse, Helmut. Sopra la formula analitica per il numero delle classi su corpi quadratici immaginari e reali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 84–95 (1951).

This paper is an all too brief report on the highly important work of the author and his pupil Curt Meyer concerning explicit formulas for the class number of a certain type of absolutely algebraic field K/P, P the field of rational numbers. It is assumed that the least normal closure N/P of K/P contains a quadratic field Ω/P so that N/Ω is abelian. Set in general $g_L=(h_LR_L)/w_L$ for a field L with class number h_L, regulator R_L, and number of roots of unity w_L; for the usefulness of this expression see the author's book reviewed below. Then let $g_{K/K_0}=g_K/g_{K_0}$ where K_0 is the maximal abelian subfield of K/P. In this manner a formula for the class number of K will be known if an intrinsic expression for g_{K/K_0} can be established since it may be assumed that g_{K_0} is given in sufficiently explicit form by the results of Kummer [see the author's book, loc. cit.]. The formula of Kummer indicates (in retrospect though) how g_K/g_{K_0} should be set up. The detailed description depends on a classification of the fields K/P according to the following principles: (α) in relation to Ω, $K\supset\Omega$ (type I) or $K\not\supset\Omega$ (type II); (β) structure of the class group G/H of K/Ω (type I) and N/Ω (type II); G/H is composed of ring classes (grade A), and (generally) G/H is composed of ray classes; (γ) is imaginary (case a), real (case b); (δ) subclasses 1a–3a of case (b) depending on the contribution of the 2 infinite primes of Ω to the conductors of the characters which determine H. For the sake of simplicity suppose that K belongs to grade A, then g_{K/K_0} is proposed as $\prod_{\chi\neq\chi_0}(-\sum_{\mathfrak{f}}\chi(\mathfrak{f}) \log F(\mathfrak{f}))$ where χ runs over all characters distinct from those characters χ_0 which belong to the maximal abelian subfield K_0, and where $\mathfrak{f}$ runs over all ring classes modulo a suitable integer f. The essential difficulty of the problem consists in finding the right kind of function $F(\mathfrak{f})$ as a class invariant which is to fit the various types arising under (α)–(δ) above. In the imaginary case (a) grade

A is settled by means of the discriminant modular function (Fueter) and B by means of a combination of said discriminant and the Weierstrass σ-function (C. Meyer). In the real case (b) a construction ("logarithmic integration" over variable parallelograms of periods) due originally to Hecke is employed for grade A, whereas for grade B subclass 1a, for example, a modified logarithmic integration using the function applied for case (a) grade B is used by C. Meyer; in this manner novel connections between imaginary and real quadratic fields are indicated. Proofs are not given in this paper; moreover the author strongly hints at the significance of the functions used by Meyer for new results on Hilbert's problem for the construction of algebraic number fields by values of suitable functions (an important class will turn out to be non-analytic, e.g., for case b, grade B, subclass 2a). *O. F. G. Schilling* (Chicago, Ill.).

★Hasse, Helmut. Über die Klassenzahl abelscher Zahlkörper. Akademie-Verlag, Berlin, 1952. xii+190 pp.

Suppose that K is a finite abelian extension of the rational number field, in the terminology of the author an absolutely abelian field, with the maximal real subfield K_0. Then, by the theorem of Kronecker, K is a subfield of a full field of roots of unity, and thus, by the class field theory, a class field belonging to a rational congruence class group H with conductor f (prime residue classes mod f, K lies in the field of all fth roots of unity). The law of reciprocity states that the characters χ of the Galois group of K may be viewed as characters of the class group mod H, and consequently, being residue class characters mod f, they determine uniquely conductors $f(\chi)$ with respect to which they become proper residue characters. Furthermore, if K is imaginary, i.e., $[K:K_0]=2$, then the group of characters $\{\chi\}$ belonging to K in the sense indicated above, has a subgroup $\{\chi_0\}$ with $\chi_0(-1)=+1$ which is precisely the character group of the real field K_0 so that the remaining characters χ are specified by $\chi(-1)=-1$; the latter are termed "characters of K/K_0". Next note that Dirichlet's theorem on units implies that the unit groups of K, K_0, respectively, have the same number n_0-1 of independent generators of infinite order; therefore the unit group of K_0 has finite index Q in the unit group of K. Then, by the analytic theory of units, $Q=2^{n_0-1}R/R_0$ for the "unit index of K/K_0", where R, R_0 denote the regulators of K, K_0 respectively. Using the theory of L-series for the characters of K, K_0 together with Hecke's product formulas (pp. 7–8) for the corresponding Gaussian sums, the author obtains by explicit evaluation of the non-principal L-series at $s=1$ the following formula for the class number h of K: $h=h_0h^*$ where

$$h_0=\frac{1}{R_0}\prod_{\chi_0\neq 1}\sum_{\pm x \bmod f(\chi_0)}(-\chi_0(x)\log|1-\zeta_{..}^x|)$$

and

$$h^*=Qw\prod_{\chi_1}(1/2f(\chi_1))\sum_{x \bmod f(\chi_1)}^{+}(-\chi_1(x)x).$$

In the expression for h_0 the summation is taken over a prime half-system x mod $f(\chi_0)$ $(0<x<(1/2)f(\chi_0))$, $\zeta_{..}$ denotes the primitive root of unity $e^{2\pi i/f(\chi_0)}$, whereas $\sum^+$ in h^* denotes summation over the system of smallest positive prime residues and w stands for the order of the group of roots of unity in K. The author poses himself the task to derive arithmetic interpretations of h_0 (the class number of K_0) and the "relative class number" h^* of K/K_0, both of which are given above in an analytical form so that their nature as positive integers is not directly recognizable. This project necessitates a recasting in modern terminology (the author does not use the concept of idèle) of major portions of early results in the theory of algebraic numbers due especially to Kummer, Kronecker, and Weber. Thus a valuable collection of special results becomes now easily accessible as an incidental by-product of the author's work which in turn is intended to set forth on one side the power of modern methods for the penetration of special problems but on the other side shows remarkably well the limitations of the tools which emphasize rather the multiplicative and ideal theoretic structure of algebraic number fields than the additive and analytic structures. Some of these limitations of class field theoretic methods are particularly noticeable in part II of the book which is devoted to the arithmetic interpretation of the formula for the class number h_0. The aim of part II ($K=K_0$, $h=h_0$, $R=R_0$ in the discussion) is to express hR or an arithmetically defined multiple thereof as the regulator of a suitable system of units in K which permits detailed handling. Letting $g_K=\prod_{p|f}\prod_{\chi\neq 1}(1-\chi(p))$ it is found that

$$g_KhR=\prod_{\chi\neq 1}\sum_{\pm x \bmod f}(-\chi(x)\log|1-\zeta_f^x|).$$

The factor g_K is interpreted as the product of the p-contributions in $\zeta(s)/\zeta_K(s)$ for $s=0$; specifically, $g_K=0$ if at least one $p|f$ has more than one factor in K, and $g_K=\prod_{p|f}n_p$ if all $p|f$ do not split and have the residue class degrees n_p, respectively. Hence $g_K=1$ if and only if K is cyclic and if some generating character χ of K has all its local components χ_p $(p|f)$ of orders equal to the degree of K. Rewriting of the summation over x in the formula for g_KhR as a summation over integral representatives s of the class group mod H yields $g_KhR=\prod_{\chi\neq 1}\sum_S(-\chi(s)\log|\lambda^S|)$ where the sum is extended over all automorphisms S (in $1-1$ correspondence with s) of the Galois group of K and where λ is the integer $\prod_{\pm a \bmod f,\, a\in H}(\zeta_{2f}^a-\zeta_{2f}^{-a})$ of the field P_{2f} of all $2f$th roots of unity. The right-hand side of the last formula for g_KhR is then recognized as the specialization $u(S)\to -\log|\lambda^S|$ of the group determinant $\prod_\chi\sum_S\chi(S)u(S)=\det(u(ST^{-1}))$ of the Galois group $S, T, \cdots$ of K; thus g_KhR turns out to be the regulator of the system of units $\eta_S=\lambda/\lambda^S$. This implies for example in the cyclic case (theorem 3_E on p. 25) that h equals $\prod_{p|f}n_p$ times the index of the "absolute values of units" of K with respect to the absolute values of the units (i.e., quotient of the corresponding regulators) which are generated by the conjugates of a special unit

$$\prod_{\pm a \bmod f,\, a\in H}[(1-\zeta_f^a)(1-\zeta_f^{-a})/(1-\zeta_f^{za})(1-\zeta_f^{-za})]^{\frac{1}{2}},$$

z an integral generator of the class group mod H. This formula contains all the number theoretic difficulties of the distribution of remainders mod f which are well known for the quadratic case. Furthermore, the result of Weber that the class number of the maximal real subfield of P_{2^ν} is odd, is generalized to certain abelian extensions so that the significance of the existence of units whose conjugates have prescribed signs ("independence of signature for units") is put into relief. In the final section of part II the author presents another transformation of hR by means of a generalized group determinant, which is adapted to the structure of K as the join of its cyclic subfields K_χ with the individual χ for generating characters. Again intricate arguments lead to indices of unit groups in K and the unknown relations between the unit groups of K and the various fields K_χ, that is, to problems which appear to evade stubbornly methods of the class field theory for example. As to the relative class number $h^*(K\supset K_0)$ it can be said

that the index computations of the class field theory lead to more complete results. Thus it is proved in part III that h_0 divides h while h^* is interpreted as the order of the subgroup of those classes in K whose norms fall into the principal class of K_0 (theorems 10, 11 on p. 49). Furthermore it follows (theorem 12 on p. 50) that h^* is divisible by 2^γ where $\gamma=\delta+q^*-1$ with δ the number of distinct prime divisors of the relative discriminant $\mathfrak{d}_0$ of K/K_0 and q^* the number of independent generators for the factor group of units in K_0 which are norm residues mod $\mathfrak{d}_0\infty$ with respect to the squares of units (theory of genera). Moreover the assumption of independence of signatures for units implies that K/K_0 is ramified. In §§20–26 (pp. 55–76) one finds a number of criteria for the determination of the value of the unit index Q occurring in h^*, which can be only 1 or 2. The approach of the author is twofold: (i) by means of the Kummer theory, and (ii) by a refined description of the latter using the theory of characters. The fact for (i) is the following theorem (theorem 15 on p. 58): Let 2^w be the precise power of 2 which divides w and set $K'=KP_{2^{w+1}}$ with the maximal real subfield K'_0. Then $Q=2$ if and only if $K'_0=K_0(\sqrt{\epsilon_0^*})$ with ϵ_0^* a unit of K_0. This criterion implies stringent conditions on the manner of ramification of K'_0/K_0. If K'_0/K_0 is "essentially ramified", i.e., the number v of distinct ramification groups for the prime divisors of 2 reaches its theoretical maximum (K'_0/K_0 is necessarily unramified for all prime divisors of odd primes), then certainly $Q=1$, whereas for "at most unessentially unramified" fields for which v is less than the theoretical maximum the alternative $Q=1$ or 2 depends on the arithmetical structure of the radicand μ_0 in $K'_0=K_0(\sqrt{\mu_0})$ with $\mu_0\cong\mathfrak{m}_0^2$ in K_0. To wit, theorem 16 on p. 59 states that $Q=1$ if and only if $\mathfrak{m}_0$ does not belong to the principal class of K_0. (It is worthwhile noting that distinctions of this type are also significant in A. Scholz's theory of metabelian extensions of algebraic number fields.) Thus theorem 16 emphasizes the distinction between the 2 types of ramifications at the prime divisors as described above; descriptions of either are next attacked by means of class field theory and the corresponding theory of characters leading, with emphasis on necessarily involved local considerations, to a comprehensive theorem (theorem 22 on p. 67) involving the precise powers of 2 which divide the conductors of the characters of K_0. However, the precise decision for $Q=2$ remains with the examination of the ideal $\mathfrak{m}_0$. These investigations lead to explicit results for special kinds of fields (§§25–26) for example (Theorem 25): If $K\supset K_0$ is abelian and "independence of signatures for units" holds in K_0, then $Q=1$. However, it should be noted (emphasizing again the complications involved in the unit groups) that the applicability of this result can in general not be inferred from the characters of K, i.e., the class field theory. The results presented in this context contain in particular the classical result that $Q=1$ or 2 for cyclotomic fields $K=P_f$ depending on whether f is a prime power or not (theorem 27 on p. 71). The results based upon the methods of class field theory, described until now, do not lend themselves in general to efforts of direct computation of h^* (as given by the original evaluations in terms of L-series and the characters χ_1 of K/K_0). In order to obtain formulas which lend themselves to explicit computations the recurrence relations of Weber for the relative class number h^* of P_{2^ν} (pp. 102–106) are generalized. For this purpose the characters χ_1 of K/K_0 are collected into Frobenius "sections". Suppose then ψ with $\psi(-1)=-1$ is a representative of a typical section, let n_ψ denote the order of ψ and N_ψ the absolute norm for the field of n_ψth roots of unity, $f(\psi)$ the corresponding conductor; all of these are invariants of the section $\{\chi_1=\psi^\mu\}$, μ a prime set of representatives mod n_ψ. Then $h^*=Qw\prod_\psi N_\psi(\theta(\psi))$ with $\theta(\psi)=(1/2f(\psi))\sum_x^+{}_{\bmod f(\psi)}(-\psi(x)x)$ in the field of n_ψth roots of unity with apparent denominators $2f(\psi)$. Congruence arguments then show (theorem 30 on p. 82) that $N_\psi(\theta(\psi))$ is an integer if ψ has composite conductor. The detailed proof of this result furnishes a formula for $\theta(\psi)$ which can be employed for numerical computations; as a matter of fact, the methods (for special cases already in the work of Kummer) of this section are aimed at such applications. In this section devoted to the determination of the actual denominator of $N_\psi(\theta(\psi))$ the author deals with detailed congruence relations often used in the early history of the theory of algebraic numbers, which were gradually pushed to the background for the recent formulation of class field theory which emphasizes in its main body of theorems the multiplicative arithmetic aspects of the underlying fields. The congruence properties together with additive analytic properties of integers are made significant here. Thus a direct proof (not using index computations of class field theory) is found for the fact that h^* is an integer (theorem 34 et sequ. on p. 94). Next the results of Weber for the relative class number of P_{2^ν} and their extension to P_{3^ν} are obtained. This section culminates with an elaboration of Kummer's results on the divisibility of the class number by 2, e.g., theorem 42 on p. 120 for a set of necessary conditions that the relative class number h^* of an imaginary field be odd, and theorem 45 on p. 124 with necessary and sufficient conditions for the oddness of the class number in imaginary cyclic fields. The appendix contains complete lists of conductors, characters, relative class numbers, and diagrams for the lattices of subfields and values of Q (with references to the pertinent theorems of the text), also other detailed explanations, for conductors up to 100. This book certainly will be extremely useful to the student of algebraic number theory who wants to come to grips with significant problems and will furnish him with a welcome crucible for the test of new thoughts on the manner in which algebraic number theory may be expanded further. Finally, it is the considered opinion of the reviewer that references to nationalities are not needed to clarify the aim of the author's valuable book. *O. F. G. Schilling* (Chicago, Ill.).

Barnes, E. S. The minimum of a factorizable bilinear form. Acta Math. 86, 323–336 (1951).

Let $B(x, y, z, t)=(\alpha x+\beta y)(\gamma z+\delta t)$ be a real bilinear form with $\Delta=\alpha\delta-\beta\gamma\neq0$, and assume that it does not represent zero integrally. Let $M(B)$ be the greatest lower bound of $|B(x, y, z, t)|$ as x, y, z, t range over all integers such that $xt-yz=1$. Set $m(B)=M(B)/|\Delta|$. Davenport and Heilbronn [Quart. J. Math. 18, 107–121 (1947); these Rev. 9, 79] proved the following: (i) $m(B)<(2^{1/2}-1)/3$ unless B is equivalent to a multiple of one of three specific forms B_1, B_2, B_3 for which $m(B_1)>m(B_2)>m(B_3)=(2^{1/2}-1)/3$, and (ii) for any $\epsilon>0$, there are uncountably many forms for which $m(B)>m(B_3)-\epsilon$. In the present paper the author gives an alternate proof of this result, using the methods of his previous paper [Proc. London Math. Soc. (3) 1, 385–414 (1951); these Rev. 13, 825]. *I. Reiner* (Urbana, Ill.).

Mordell, Louis Joel. The product of n homogeneous linear forms. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 12–23 (1951).

An expository article discussing many of the important results in the geometry of numbers; especially some of the

recent results obtained by the English school. The central problem of the discussion is that of estimating the greatest lower bound for all integral $x_1, \cdots, x_n$ of $|L_1L_2\cdots L_n|/D$, where $L_i=\sum_{j=1}^n l_{ij}x_j$ are linear forms with complex coefficients and D is the absolute value of the determinant $|l_{ij}|$.
E. G. Straus (Los Angeles, Calif.).

Rogers, C. A. Indefinite quadratic forms in n variables. J. London Math. Soc. **27**, 314–319 (1952).

Let $Q(u_1, \cdots, u_n)$ be an indefinite quadratic form in n variables with real coefficients and with non-zero determinant D. A theorem of Blaney [same J. **23**, 153–160 (1948); these Rev. **10**, 511] asserts that for any real numbers $\alpha_1, \cdots, \alpha_n$ integers $u_1, \cdots, u_n$ exist for which

$$(1) \qquad |Q(u_1+\alpha_1, \cdots, u_n+\alpha_n)| \leqq c_n|D|^{1/n}, \quad c_n=2^{n-2}.$$

In order to improve on this result the author considers linear forms

$$x_i=a_i^{(1)}u_1+\cdots+a_i^{(n)}u_n \quad (i=1, \cdots, n),$$

such that

$$Q(u_1, \cdots, u_n)=x_1^2+\cdots+x_r^2-x_{r+1}^2-\cdots-x_n^2$$

and using a fundamental lemma of Mahler [Proc. Roy. Soc. London. Ser. A. **187**, 151–187 (1946), p. 156; these Rev. **8**, 195] reduces the problem to the case that (I) $D=1$, (II) the lattice Λ, consisting of the points $(x_1, \cdots, x_n)$ which arise from integral values of $u_1, \cdots, u_n$, contains n independent points on the boundary of the sphere $x_1^2+\cdots+x_n^2 \leqq \{m(\Lambda)\}^2$, where $m(\Lambda)$ denotes the smallest distance between two different points of Λ. This leads to the result that in (1) one may take $c_n=\frac{1}{4}n^2\gamma_n$, where γ_n is Hermite's constant (for which we have lim sup $\gamma_n/n=1/\pi e$). In a note added in proof the author communicates an argument of Davenport showing that one may even take $c_n=\frac{1}{4}n\gamma_n$.
J. F. Koksma (Amsterdam).

Varnavides, P. The Minkowski constant of the form x^2-11y^2. Bull. Soc. Math. Grèce **26**, 14–23 (1952). (Greek summary)

The Minkowski constant $m(f)$ of the quadratic form $f(x, y)=ax^2+bxy+cy^2$ with discriminant $d>0$ is the lower bound of numbers λ such that for arbitrary real x_0, y_0 there exist integers x, y such that $f(x+x_0, y+y_0) \leqq \lambda\sqrt{d}$. This constant was determined for x^2-2y^2 by Davenport [Nederl. Akad. Wetensch., Proc. **49**, 815–821 (1946); these Rev. **8**, 444] and for x^2-7y^2 by the author [Proc. Roy. Soc. London. Ser. A. **197**, 256–268 (1949); these Rev. **10**, 682]. The form x^2-11y^2 is treated in this paper with the result $m(f)=19/44\sqrt{11}$, an improvement on Perron's inequality $m(f) \leqq 1/2\sqrt{11}$. The method is similar to that used in the 1949 paper but deals with the field $k(\sqrt{11})$ instead of $k(\sqrt{7})$.
T. M. Apostol (Pasadena, Calif.).

Malyšev, A. V. On the Minkowski-Hlawka theorem concerning a star body. Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 168–171 (1952). (Russian)

The author extends the Minkowski-Hlawka theorem on bounded star bodies to unbounded star bodies. I.e., he proves that if the volume of an n-dimensional symmetric star body, centred at the origin, is less than $2\zeta(n)$, then it is possible to find a unimodular transformation transforming the body into one containing no lattice point other than the origin. This is proved by expanding the body about one axis Ox_1 and contracting it about the others so that the sum of the areas of sections at distance unity apart and perpendicular to Ox_1 is approximately equal to the volume. A line not in the plane $x_1=0$ is then chosen as the new x_1 axis in such a way that the new body contains no lattice point other than those in the plane $x_1=0$. This process is carried out for each of the n axes, the body being finally transformed into one containing no lattice-point other than the origin. Similar extensions of the Minkowski-Hlawka theorem valid under different or additional conditions have been mentioned by Siegel [Ann. of Math. **46**, 340–347 (1945), p. 346; these Rev. **6**, 257] and Rogers [ibid. **48**, 994–1002 (1947), p. 1000; these Rev. **9**, 270].
R. A. Rankin (Birmingham).

***Korobov, N. M. Fractional parts of exponential functions.** Trudy Mat. Inst. Steklov., v. 38, pp. 87–96. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

Let $\varphi(x)$ be any completely uniformly distributed function, in the sense of the author's earlier paper [Uspehi Matem. Nauk **4**, no. 1 (29), 189–190 (1949); these Rev. **11**, 231], for which the system of functions $\varphi(x+1)$, $\varphi(x+2), \cdots, \varphi(x+s)$ retains the property of uniform distribution under any linear transformation $x=\lambda y$. Define α_ν $(\nu=1, 2, \cdots, n)$ by

$$\alpha_\nu=\sum_{k=1}^{\infty}[q_\nu\{\varphi(kn+\nu)\}]/q_\nu^k,$$

where the q_ν are any given integers greater than unity. Then the author proves that the function $F(x)=\alpha_1q_1^x+\cdots+\alpha_nq_n^x$ is uniformly distributed. It is also proved that the function $\alpha f(x)a^x$ is uniformly distributed when a is an integer greater than unity, $f(x)$ a polynomial of positive degree with integral coefficients and α is defined as a rapidly converging power series in $1/a$ with coefficients of a special form. Finally, the author considers the exponential sum $S=\sum_{x=1}^P e^{2\pi i\alpha q^x}$, where q is an integer greater than unity and $0<\alpha<1$. Since $\int_0^1|S|^2d\alpha=P$, it follows that, corresponding to every $\epsilon>0$, we can find a sufficiently large $C=C(\epsilon)$ such that $|S|<C\sqrt{P}$ for every α in $(0, 1)$ except possibly for a set of measure less than ϵ. By means of his theory of normal periodic systems [Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 17–46 (1951); these Rev. **13**, 213] the author constructs a wide class of numbers α for which the inequality $|S|<(4\pi+1)q\sqrt{P}$ holds.
R. A. Rankin (Birmingham).

Korobov, N. M., and Postnikov, A. G. Some general theorems on the uniform distribution of fractional parts. Doklady Akad. Nauk SSSR (N.S.) **84**, 217–220 (1952). (Russian)

It is proved that if for every positive integer h the function $F(x+h)-F(x)$ is uniformly distributed, then, for every pair of integers λ, μ, the function $F(\lambda x+\mu)$ is also uniformly distributed. This is a generalisation of a result of van der Corput [Acta Math. **56**, 373–456 (1931)]. Several applications of this result are given. It is also used to prove a general theorem on trigonometrical sums from which the following corollary is deduced. Let α be a real number for which $|\sum_{x=1}^P e^{2\pi i m\alpha q^x}|<cm\sqrt{P}$. Here the notation is that of the paper reviewed above, m is a positive integer, and the methods of that paper can be used to construct numbers α satisfying this inequality. Then it follows that for any positive integer

$$\left|\sum_{x=1}^{P} e^{2\pi i m\alpha q^{\lambda x}}\right|<c(m)\lambda\frac{P}{\sqrt{\log P}}.$$

R. A. Rankin (Birmingham).

Korobov, N. M. Some many dimensional problems of the theory of Diophantine approximations. Doklady Akad. Nauk SSSR (N.S.) 84, 13–16 (1952). (Russian)

In this paper the author constructs, with the aid of his theory of normal periodic systems, s real numbers $\alpha_1, \alpha_2, \cdots, \alpha_s$ such that the function $(m_1\alpha_1+m_2\alpha_2+\cdots+m_s\alpha_s)q^x$ is uniformly distributed; i.e., the system of functions $\alpha_1 q^x, \cdots, \alpha_s q^x$ is uniformly distributed in s-dimensional space. Here q is an integer greater than unity and $m_1, m_2, \cdots, m_s$ are integers not all zero. Two further theorems concerning systems of functions uniformly distributed in s-dimensional space, which are analogous to the first two results mentioned in the second preceding review, are given; it is stated that they may be proved by similar methods. *R. A. Rankin.*

Korobov, N. M. On normal periodic systems. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 211–216 (1952). (Russian)

The author generalises his theory of normal periodic systems [same Izvestiya 15, 17–46 (1951); these Rev. 13, 213] in the following way. Let n, q, τ, δ_ν be integers satisfying $n\geqq 1$, $q\geqq 2$, $\tau\geqq n$, $0\leqq\delta_\nu\leqq q-1$. Here the δ_ν may be regarded as digits in the scale of q. Denote by E_n any set of τ numbers each containing n digits δ_ν; the numbers of E_n need not be distinct. Then he says that a normal periodic system is possible in the set E_n if there exists a sequence of digits

$$\delta_1\delta_2\delta_3\cdots\delta_{n-1}\delta_n\delta_{n+1}\cdots\delta_\tau\delta_1\delta_2\cdots\delta_{n-1}$$

with the property that E_n coincides with the set of τ numbers of n digits which can be formed from successive batches of n digits of this sequence. When E_n is the set of q^n n-digited numbers in the scale of q, this definition coincides with that given in his earlier paper. A number of $n-1$ digits $\beta_1\beta_2\cdots\beta_{n-1}$ is said to occur in E_n if E_n contains a number of one of the forms $\beta_1\beta_2\cdots\beta_{n-1}\beta$, $\beta'\beta_1\beta_2\cdots\beta_{n-1}$. The following theorem is proved. A normal periodic system is possible in a set E_n if and only if the following two conditions are both satisfied: (i) For every number of $n-1$ digits $\beta_1\beta_2\cdots\beta_{n-1}$ occurring in E_n the numbers of numbers of the form $\beta_1\beta_2\cdots\beta_{n-1}\beta$ and $\beta'\beta_1\beta_2\cdots\beta_{n-1}$ contained in E_n are equal; (ii) for every partition of E_n into two non-null parts E_n' and E_n'' there can be found at least one number of $n-1$ digits which occurs in each of these parts. The proof of the necessity of these conditions is straightforward. In order to prove them sufficient a method is given which actually constructs a normal periodic system in the set E_n. Various examples of the theorem are given and the existence of normal periodic systems in the earlier sense follows as a simple corollary. The author also investigates the set E_n of numbers defined as follows: Let u_n $(n=0, 1, 2, \cdots)$ denote the Fibonacci numbers, and write $\delta_1\delta_2\cdots\delta_n$ for the number $\delta_1 u_{n-1}+\delta_2 u_{n-2}+\cdots+\delta_n u_0$ where $q=2$ and no two neighbouring digits δ_i, δ_{i+1} are both unity. Every positive integer can be represented uniquely in this way for a suitable n. Then E_n denotes the set of all n-digited numbers of this form. It is easily shown that normal periodic systems do not exist in E_n. However, if E_n is modified by the omission of those numbers which contain 1 as first and last digit, then it is proved that normal periodic systems exist in the new set so formed. *R. A. Rankin.*

ANALYSIS

Fujinaka, Hiroshi. On the solution of the integral inequality $xu(x)\leqq\int_0^x(\nu+\epsilon(t))u(t)dt$. Math. Japonicae 2, 143–145 (1952).

It is demonstrated that if $\epsilon(x)\geqq 0$ on I: $0\leqq x\leqq r$ with $\int_0^r\epsilon(t)dt/t<\infty$, then the only solution of the inequality in the title which is continuous on I and satisfies the condition $0\leqq u(x)=o(x^{\nu-1})$ is $u(x)\equiv 0$. Applied to differential equations, this yields the uniqueness of the solution of a differential equation $y'=f(x, y)$, through (a, b) if $f(x, y)$ is continuous in a vicinity of (a, b), and satisfies there the inequality $|f(x, \bar y)-f(x, y)|<(1+\epsilon(x-a)|\bar y-y|/|x-a|$, a result due to Shimizu [Proc. Imp. Acad. Tokyo 4, 326–329 (1928)], generalizing a condition of Nagumo.

T. H. Hildebrandt (Ann Arbor, Mich.).

Picone, Mauro. Vedute generali sull'interpolazione e qualche loro conseguenza. Ann. Scuola Norm. Super. Pisa (3) 5, 193–244 (1951).

Soit

$$E=\sum a_{i_1\cdots i_r}\frac{\partial^{i_1+\cdots+i_r}}{\partial x_1^{i_1}\cdots\partial x_r^{i_r}}\quad (i_1+\cdots+i_r\leqq k)$$

un opérateur linéaire, les $a(P)$ étant définis et continus dans un domaine T et sur sa frontière $\mathfrak{F}T$; on désigne par $\{u\}$ l'espace des fonctions telles que $E(u)$ est continu dans T, par $L_h(u)$ des opérateurs linéaires définis dans $\{u\}$ tels que: (1) le problème

$$E(u)=e(P),\quad L_h(u)=l_h(P)\quad (h=1, 2, \cdots, m)$$

ait une solution et une seule dans $\{u\}$, lorsque $e(P)$ et le vecteur $l_h(P)$ appartiennent à certains espaces précisés; (2) pour $l_h(P)=0$, cette solution est donnée par

$$u(P)=\int_T G(P, Q)e(Q)dT$$

(G fonction de Green du problème).

Les méthodes d'interpolation employées pratiquement consistent à remplacer $f(P)$ par la solution u_f de

$$E(u_f)=0,\quad L_h(u_f)=L_h(f).$$

L'erreur est alors $\int_T G(P, Q)E(f)dT$; et pour une opération linéaire Λ elle est $\int_T\Gamma_\Lambda(Q)E(f)dT$ $(\Gamma_\Lambda(Q)=\Lambda[G(P, Q)])$. Les formules permettent d'évaluer l'erreur lorsque G ou Γ_Λ peuvent être effectivement calculés. Etude du cas où $E=\Delta=\sum_1^r\partial^2/\partial x_s^2$, $L(u)=u(P)$ sur $\mathfrak{F}T$ (et de $E=\Delta^s$). Expression, puis évaluation effective de l'erreur, dans le cas des fonctions d'une variable réelle définies sur un segment, soit pour les fonctions, soit pour les formules de quadrature dans l'interpolation par polynomes (avec des conditions très variés), ou dans le cas des fonctions périodiques approchées par des polynomes trigonométriques, ou dans le cas des fonctions holomorphes dans certains domaines.

Dans le cas de deux variables, pour le calcul approché d'une intégrale double dans un rectangle, l'auteur propose une formule contenant les intégrales simples de la fonction le long des côtés du rectangle et le long des parallèles aux côtés passant par le centre, ainsi que les valeurs de la fonction aux 9 sommets de cette figure. Cette formule, qui est l'extension naturelle de celle de Cavalieri-Simpson dans le cas d'une variable, est, d'après l'auteur, destinée à rendre les mêmes services que celle-ci. *J. Favard* (Paris).

Ahiezer, N. I. On solutions of the power problem of moments in the indeterminate case. Učenye Zapiski Har'kov. Gos. Univ. 28, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) 20, 99–106 (1950). (Russian)

The author gives a new proof of Nevanlinna's characterization of the general solution of an indeterminate Hamburger moment problem [see Shohat and Tamarkin, The problem of moments, American Mathematical Society, New York, 1943, p. 57; these Rev. 5, 5]. The proof does not depend on continued fractions or on operator theory, but does depend on an earlier paper of the author [Uspehi Matem. Nauk 9, 126–156 (1941); these Rev. 3, 110] for facts about Jacobi matrices and definitions of relevant quantities. The theorem follows readily from the following generalization of Pick's theorem on functions with positive imaginary part. In order that there exist a function $w(z)$ $(\Im(z)>0)$ of the form $\int_{-\infty}^{\infty}(\lambda-z)^{-1}d\psi(\lambda)$, where ψ is a solution of the Hamburger moment problem, such that $w(z_k)=w_k$ $(\Im(z_k)>0)$, it is necessary and sufficient that the Hermitian forms $\sum\sum_1^n\{w_\alpha, w_\beta\}\xi_\alpha\bar{\xi}_\beta$ are non-negative; here

$$\{w_\alpha, w_\beta\}=\frac{w_\alpha-\bar{w}_\beta}{z_\alpha-\bar{z}_\beta}-\sum_{k=0}^{\infty}[P_k(z_\alpha)w_\alpha+Q_k(z_\alpha)]\overline{[P_k(z_\beta)w_\beta+Q_k(z_\beta)]}$$

and P_k, Q_k are the polynomials denoted by ω_k, χ_k in Shohat and Tamarkin's book (of course defined in a different way by Ahiezer).

R. P. Boas, Jr. (Evanston, Ill.).

Glazman, I. M. On a class of solutions of the classical problem of moments. Učenye Zapiski Har'kov. Gos. Univ. 28, Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) 20, 95–98 (1950). (Russian)

According to Nevanlinna's theorem [see the preceding review] the solutions of the indeterminate moment problem

$$\mu_k=\int_{-\infty}^{\infty}t^k d\psi(t) \tag{1}$$

are given by the formula

$$\int_{-\infty}^{\infty}\frac{d\psi(t)}{t-z}=\frac{A(z)+C(z)\tau(z)}{B(z)+D(z)\tau(z)}, \tag{2}$$

where A, B, C, D are certain entire functions and $\tau(z)$ is regular in the upper half-plane and has its imaginary part non-negative (N-function). A solution $\psi_\tau(t)$ corresponding to $\tau(z)=\tau$, a real constant (or $+\infty$) is called canonical. An N-function $F(z)$ is called of resolvent type if it has the form $\int_{-\infty}^{\infty}(t-z)^{-1}d\mu(t)$ with $\mu(t)$ nondecreasing and of total variation 1 (or equivalently if $\lim_{y\to+\infty}iyF(iy)=-1$). The collection of all solutions $\psi(t)$ of (1) of the form $\psi(t)=\int_{-\infty}^{\infty}\psi_\tau(t)d\mu(\tau)$, with $\mu(t)$ as above, is called the convex envelope of the canonical solutions. The author's theorem is that $\psi(t)$ belongs to the convex envelope of the canonical solutions if and only if $\tau(z)$ in (2) has the form $\tau(z)=1/F(v)+v$, where $F(v)$ is an N-function of v of resolvent type and $v=-B(z)/D(z)$. He also indicates a generalization to Hermitian operators of deficiency index (n, n).

R. P. Boas, Jr. (Evanston, Ill.).

Calculus

Bückner, Hans. A formula for an integral occurring in the theory of linear servomechanisms and control-systems. Quart. Appl. Math. 10, 205–213 (1952).

Assume all the zeros of the polynomial

$$f(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n$$

with real coefficients have negative real parts ($f(z)$ is a "Hurwitz polynomial"). The convergent integral $Y=\int_0^\infty y^2(t)dt$, where $y(t)$ is a solution of

$$f(p)y=0,\quad p^k y(0)=q_k\quad (k=0, 1, \cdots, n-1),\quad p=d/dt,$$

is used in the analysis of servomechanisms. The author develops an explicit formula for Y in terms of the a_i and q_k. To derive this formula a new algorithm for reducing a Hurwitz polynomial to one of lower degree is established.

M. Golomb (Lafayette, Ind.).

***Hardy, G. H. A course of pure mathematics.** 10th ed. Cambridge, at the University Press, 1952. xii+509 pp. $4.75.

This edition differs from the 9th in the addition of an index and new proofs of two theorems.

***Smirnov, V. I. Kurs vysšeĭ matematiki. Tom IV. [A course of higher mathematics. Vol. IV.]** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 804 pp. 23.45 rubles.

For a review of the first edition see these Rev. 6, 42. For this edition much of the material has been rewritten and the order of presentation altered. Table of contents: I. Integral equations. II. The calculus of variations. III. The general theory of partial differential equations. IV. Boundary problems.

***Angot, André. Compléments de mathématiques à l'usage des ingénieurs de l'électrotechnique et des télécommunications.** 2ème éd. Editions de la Revue d'Optique, Paris, 1952. viii+688 pp.

1) Quantités complexes et applications. 2) Série de Fourier; intégrale de Fourier. 3) Calcul vectoriel. 4) Calcul matriciel. 5) Notions élémentaires sur les tenseurs; applications. 6) Méthodes d'intégration des équations différentielles. 7) Notions sur quelques fonctions usuelles. 8) Calcul symbolique. 9) Calcul des probabilités; applications.

Table of contents.

***Lotze, Alfred. Vektor- und Affinor-Analysis.** Verlag von R. Oldenbourg, München, 1950. 276 pp.

A useful expository work. The chapter titles are: (I) Vector and affinor algebra; (II) vector and affinor analysis; (III) application to differential geometry; (IV) application to mechanics (including deformable bodies); (V) application to the electromagnetic field. There is an appendix on 4-space vectors with application to electrodynamics of the restricted theory of relativity.

L. M. Milne-Thomson.

***Ryšavý, Vladimír. Vektory a tensory. [Vectors and tensors.]** Jednota Československých Matematiků a Fysiků, Prague, 1949. 121 pp. Kčs. 44.00.

Devidé, Vladimir. Beweis einiger Sätze der Vektorrechnung mittels der Quaternion-Algebra. Bull. Soc. Math. Phys. Serbie 3, nos. 3–4, 49–50 (1951). (Serbo-Croatian. German summary)

Backes, F. Sur un théorème d'analyse vectorielle. Mathesis 61, 87–89 (1952).

Theory of Sets, Theory of Functions of Real Variables

Sierpiński, Wacław. Sur un problème de M. Zarankiewicz. Soc. Sci. Lett. Varsovie. C. R. Cl. III, Sci. Math. Phys. 42 (1949), 1–3 (1952). (French. Polish summary)

The author shows that for every ordinal number α, where $1<\alpha<\omega_1$, there exists a linear ambiguous set of class α [see, e.g., Kuratowski, Topologie I, 2nd ed., Warszawa-Wrocław, 1948, p. 254; these Rev. 10, 389] which is not of class $<\alpha$ in any interval. This solves a problem posed by Zarankiewicz [Wiadom. Mat. 30, 127–132 (1928)]. *F. Bagemihl.*

Novák, Josef. On some ordered continua of power $2^{\aleph_0}$ containing a dense subset of power $\aleph_1$. Czechoslovak Math. J. 1(76), 63–79 (1951) = Čehoslovack. Mat. Ž. 1(76), 77–97 (1951).

Mišík, L. On one ordered continuum. Czechoslovak Math. J. 1(76), 81–86 (1951) = Čehoslovack. Mat. Ž. 1(76), 99–105 (1951).

Novotný, Miroslav. Construction de certains continus ordonnés de puissance $2^{\aleph_0}$. Czechoslovak Math. J. 1(76), 87–95 (1951) = Čehoslovack. Mat. Ž. 1(76), 107–116 (1951).

Novák identifies the neighboring elements of the lexicographically ordered set of all sequences, of type ω_1, of zeros and ones, to obtain a continuous set Q. He then constructs six continuous ordered sets $\mathfrak{P}_1(c_{00}, c_{01})$, $\mathfrak{P}_2(c_{00}, c_{10})$, $\mathfrak{P}_3(c_{00}, c_{01}, c_{10}, c_{11})$, $\mathfrak{P}_4(c_{00}, c_{01}, c_{11})$, $\mathfrak{P}_5(c_{00}, c_{10}, c_{11})$, $\mathfrak{P}_6(c_{00}, c_{11})$ (the set of point characters of each $\mathfrak{P}_k$ (minus its endpoints) follows it in parentheses), each of which is a set of mutually exclusive (closed) intervals and single points of Q which exhaust Q, ordered in the natural way. For every k $(1\leqq k\leqq 6)$: (1) $|\mathfrak{P}_k|=2^{\aleph_0}$; (2) $\mathfrak{P}_k$ contains a subset which is dense in $\mathfrak{P}_k$ and of power $\aleph_1$, but contains no such subset of power $\aleph_0$; (3) every interval of $\mathfrak{P}_k$ contains a subinterval which is similar to $\mathfrak{P}_k$; (4) $\mathfrak{P}_k$ is similar to a subset of $\mathfrak{P}_3$; (5) every nonenumerable set of mutually exclusive intervals of $\mathfrak{P}_k$ contains a nonenumerable subset of intervals whose left endpoints form an increasing or a decreasing sequence of elements of $\mathfrak{P}_k$; (6) a continuous ordered set S possessing the Souslin property contains an enumerable subset which is dense in S, if and only if S is similar to a subset of $\mathfrak{P}_k$. [It is to be remarked, in connection with the last paragraph on p. 79, that $P(c_{01}, c_{10}, c_{11})$ could exist only if $2^{\aleph_0}>\aleph_1$; see Hausdorff, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. 59, 84–159 (1907), p. 85.] Mišík, using Novák's method, constructs a continuous ordered set $\mathfrak{P}_7(c_{00}, c_{01}, c_{10})$ such that (1), $\cdots$, (6) hold for $k=7$, and thereby solves a problem posed by Novák (p. 79). Novotný shows that $\mathfrak{P}_3$ is similar to the "ultracontinuum" constructed by Bernstein [Math. Ann. 61, 117–155 (1905), p. 152] (which proves a conjecture of Novák, p. 72, footnote), and indicates a construction of $\mathfrak{P}_1, \cdots, \mathfrak{P}_7$ which is analogous to the method of Bernstein just cited. Novotný also obtains seven continuous ordered sets $\mathfrak{P}_1', \cdots, \mathfrak{P}_7'$, each of power $2^{\aleph_1}$, such that $\mathfrak{P}_k'$ $(1\leqq k\leqq 7)$ contains a subset which is dense in $\mathfrak{P}_k'$ and of power $2^{\aleph_0}$, but contains no such subset of smaller power, and he derives some further properties (like ones derived for $\mathfrak{P}_1, \cdots, \mathfrak{P}_7$) of these sets. [The Lemma on p. 90 is a special case of a theorem proved by Hausdorff, Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. 58, 106–169 (1906), p. 140. The present authors do not use the words "continuum" and "ultracontinuum" in the sense recommended by Hausdorff, ibid., p. 166.] *F. Bagemihl.*

Kurepa, Gjuro. On a definition and notation of matrices. On a kind of switch matrices. Bull. Soc. Math. Phys. Serbie 4, nos. 1–2, 1–7 (1952). (Serbo-Croatian summary)

As an illustration of the notion of mapping, the author defines, in the natural way, a matrix of α rows and β columns, where α and β are arbitrary nonzero ordinal numbers. A matrix, every element of which belongs to the set $\{0, 1\}$, is called a switch matrix, and this kind of matrix is discussed briefly. [The "Lemma" on page 3 is false, as the following counterexample shows: $Sf=(\omega, \omega+1)$, $f(n+1, n)=1$ $(n=0, 1, 2, \cdots)$, $f(0, \omega)=1$, $f(\xi, \eta)=0$ for every other ordered pair (ξ, η) with $\xi<\omega$, $\eta<\omega+1$.] *F. Bagemihl.*

Eyraud, Henri. Le théorème du continu. Cahiers Rhodaniens no. 3, 22 pp. (1951).

This paper, which opens with the statement that "the demonstration of the continuum theorem is neither long nor particularly arduous", contains a futile attempt to prove that $2^{\aleph_0}=\aleph_1$. The author begs the question on page 18, lines 23–25. (He also fails to prove the assertion made on page 21, lines 23–26.) *F. Bagemihl* (Rochester, N. Y.).

Obreanu, Filip. Zorn's theorem. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 687–692 (1949). (Romanian. Russian and French summaries)

Let T be a set with a partial order relation $<$ which is transitive but not necessarily reflexive or antisymmetric. For $A\subset T$, write $M(A)$ for $E[x; x\in T, x>a$ for all $a\in A]$ and $I(A)$ for $E[x; x\in T, x<a$ for all $a\in A]$. Write $\sup A$ for $M(A)\cap I(M(A))$. The set T is said to be inductive if for every completely ordered subset L of T, $\sup L\neq 0$. For every x in an inductively partially ordered set T as described, there exists a maximal y in T such that $x<y$. This is of course a form of Zorn's principle; the proof uses the axiom of choice. *E. Hewitt* (Seattle, Wash.).

Ellis, J. W. A general set-separation theorem. Duke Math. J. 19, 417–421 (1952). 2^S 2^S

Let S be a set and φ and ψ functions on 2^S to 2^S. If φ and ψ both satisfy (P1)–(P4) below, and jointly satisfy (P5), then for each pair A, B of subsets of S such that $\varphi A=A$, $\psi B=B$, and $A\cap B=\Lambda$, there are sets $A'\supset A$ and $B'\supset B$ such that $\varphi A'=A'$, $\varphi B'=B'$, $A'\cap B'=\Lambda$, and $A'\cup B'=S$. (P1) $E\subset S$ implies $\varphi E\supset E$. (P2) $\varphi\varphi=\varphi$. (P3) $E\subset S$ implies $\varphi E=\sigma\{\varphi F|\text{finite } F\subset E\}$. (P4) finite $F\subset S\ \&\ p\in S$ implies $\varphi(F\cup\{p\})=\sigma\{\varphi\{a, b\}|a\in\varphi F\}$. (P5) $a\in\psi\{b, p\}\ \&\ c\in\varphi\{d, p\}$ imply $\varphi\{a, d\}\cap\psi\{b, c\}\neq\Lambda$. (Here σM denotes the union of all sets in the family M.) As applications are obtained results of Stone on ideals in distributive lattices, of Kakutani and Tukey on convex sets in linear spaces, and of the reviewer on dense convex cones. *V. L. Klee, Jr.*

Ribeiro de Albuquerque, J. Théorie des ensembles projectifs. Portugaliae Math. 11, 11–33 (1952).

This is a résumé of the author's thesis presented to the Faculty of Sciences at Lisbon. However, proofs are given for all but the last result actually stated. The paper treats of the "analytic" sets generated by Suslin's operation (A) carried out on an initial family. The process is then repeated on those analytic sets whose complements are also analytic.

The novelty lies in the fact that topological concepts are not admixed. In some places (p. 28) the meanings are a bit obscure but apparently separation theorems for disjoint analytic sets are obtained which seem to reduce to the usual ones in separable metric spaces. Some problems stated by Lusin are solved. *R. Arens* (Los Angeles, Calif.).

Ribeiro de Albuquerque, J. Un théorème sur les ensembles criblés. Portugaliae Math. 11, 95–103 (1952).

There is given a corrected proof of the following theorem presented in the paper reviewed above: Let C be a rational sieve (crible) extracted from F (a family of sets) which sifts E. Then C is bounded on every set E^* disjoint from E, which is also sifted by a rational sieve C^* extracted from F. *R. Arens* (Los Angeles, Calif.).

Marczewski, E., and Sikorski, R. On isomorphism types of measure algebras. Fund. Math. 38, 92–98 (1951).

A measure μ defined on a Boolean σ-algebra A is said to be strictly positive if $\mu(a)>0$ for every nonzero element $a \in A$. Two measures are said to be isomorphic if there is a measure-preserving isomorphism between the Boolean σ-algebras on which they are defined. It is proved that if μ and ν are strictly positive measures defined on Boolean σ-algebras A and B respectively, and if A_0 and B_0 are σ-subalgebras of A and B respectively such that μ is isomorphic to the restriction of ν to B_0 and ν is isomorphic to the restriction of μ to A_0, then the measures μ and ν are isomorphic. *L. H. Loomis* (Cambridge, Mass.).

Nunke, R. J., and Savage, L. J. On the set of values of a nonatomic, finitely additive, finite measure. Proc. Amer. Math. Soc. 3, 217–218 (1952).

The following theorem is proved. If a Boolean algebra B carries any finitely additive, nonatomic measures at all, then it carries one such measure, say m, such that $m(e)=4$, where e is the identity of B, and such that none of the values of m lie in the interval $(1, 3)$. *L. H. Loomis.*

Hadwiger, H., und Kirsch, A. Zerlegungsinvarianz des Integrals und absolute Integrierbarkeit. Portugaliae Math. 11, 57–67 (1952).

The main point of the paper is to extend Tarski's concept of absolute measurability for sets to a concept of absolute integrability for functions. Let $\mathfrak{B}$ be the set of all real-valued functions on the real line such that if $F \in \mathfrak{B}$, then F is bounded and F vanishes outside a finite interval. Let E be the characteristic function of the interval $0 \leqq x < 1$. An integration system $(\mathfrak{F}, J)$ is a translation-invariant linear subclass $\mathfrak{F}$ of $\mathfrak{B}$ such that $E \in \mathfrak{F}$, and a translation-invariant, positive, and normalized linear functional J on $\mathfrak{F}$. After defining decomposition-equivalence $(F \sim G)$ and decomposition-ordering $(F \sim\leqq G)$ in the natural manner, and studying the elementary properties of these concepts, the authors define $\bar{T}(F)=\inf\{k: F \sim\leqq kE\}$ and $\underline{T}(F)=\sup\{k: kE \sim\leqq F\}$ and call a function F in $\mathfrak{B}$ absolutely integrable, with absolute integral $T(F)$, if $\bar{T}(F)=\underline{T}(F)$ $(=T(F))$. The set $\mathfrak{T}$ of all absolutely integrable functions, with the absolute integral T, is an integration system and, moreover, $\mathfrak{T}$ is decomposition-invariant. If $(\mathfrak{F}, J)$ is an integration system and $F \in \mathfrak{F}$, then $\underline{T}(F) \leqq J(F) \leqq \bar{T}(F)$. Conversely, if $F \in \mathfrak{B}$, then there exists an integration system $(\mathfrak{F}, J)$ such that $F \in \mathfrak{F}$ and such that $J(F)$ has any prescribed value between $\underline{T}(F)$ and $\bar{T}(F)$. The set $\mathfrak{T}$ contains the set $\mathfrak{R}$ of Riemann-integrable functions but is incomparable with the set $\mathfrak{L}$ of Lebesgue-integrable functions. *P. R. Halmos* (Chicago, Ill.).

***Gomes, Ruy Luís. Integral de Lebesgue-Stieltjes num espaço localmente compacto. I. [The Lebesgue-Stieltjes integral in a locally compact space. I.]** Cadernos de Análise Geral, no. 21, Junta de Investigação Matemática, Porto, 1952. ix+94 pp.

The definition of summable functions and of their integral for a given measure on a locally compact space E, which is given in this book, is a variant of the now well-known "functional" method, stemming from the work of P. J. Daniell [cf. M. H. Stone, Proc. Nat. Acad. Sci. U. S. A. 34, 336–342, 447–455, 483–490 (1948); 35, 50–58 (1949); these Rev. 10, 24, 107, 239, 360; and N. Bourbaki, Actualités Sci. Ind., no. 1175, Hermann, Paris, 1952]. On the set F^+ of all functions $f \geqq 0$ defined in E (finite or not), the author considers "intervals" $[h, g]$, i.e., the set of all $f \in F^+$ such that $h \leqq f \leqq g$, where g is lower semi-continuous, and h finite, upper semi-continuous, and of compact carrier. This defines a system of neighborhoods for a topology on F^+; if L^+ is the subset of F^+ consisting of (positive) continuous functions with compact carrier, the integral of $f \in L^+$ is shown to be continuous on L^+ and is extended by continuity to all functions of F^+ where it has a limit. Additivity and the usual Lebesgue convergence theorems follow as usual. Then measurability of functions is defined in the classical way (a measurable set being a denumerable union of sets whose characteristic function is summable, and measurable functions being defined by means of measurable sets), and the book ends with the classical theorems of Egoroff, Lusin, and Riesz.

The example given on p. 4 does not make sense unless $a=c$ or $b=d$. Egoroff's theorem is incorrectly stated (on p. 65): the condition that E be measurable is of course not sufficient, as the sequence of the characteristic functions of the intervals $[-n, +n]$ on the real line shows at once; with the notation of the author, the measure of $E_n(\delta)$ in that case is always $+\infty$ for $\delta<1$. Owing to that error, the proof of Lusin's theorem as stated on p. 66, is incorrect. With the exception of these points, the book is very clearly written, and proves that it is possible to give in a short space the essentials about the Lebesgue integral in its most general form, without going through the dreary and useless developments of the classical Carathéodory theory. *J. Dieudonné* (Ann Arbor, Mich.).

Taylor, S. J. On Cartesian product sets. J. London Math. Soc. 27, 295–304 (1952).

This note is related to the problem of determining conditions under which $\dim A \times B = \dim A + \dim B$, dimension being taken in the Besicovitch sense. The author shows that, given any linear set A of zero α-measure, there exists a perfect linear set B such that $A \times B$ has zero α-measure. This result is used to prove that, if A has Besicovitch dimension α, $0 \leqq \alpha \leqq 1$, (but not necessarily positive or finite α-measure), then a perfect zero-dimensional linear set B exists such that $\dim A \times B = \dim A$. Finally, it is shown that if there exists a single zero-dimensional linear set B with the power of the continuum such that, for any linear set A, the α-measure of $A \times B$ is zero whenever the α-measure of A is zero, then B cannot contain a perfect subset, and hence cannot be a Borel set, or even an analytic set. *L. H. Loomis.*

Masani, P. What is a function? Math. Student 19 (1951), 81–101 (1952).

Aubert, K. E. Continuity and discrete functions. Norsk Mat. Tidsskr. 34, 33–41 (1952). (Norwegian)

An expository discussion of the notion of continuity for functions. *E. R. Lorch* (New York, N. Y.).

Bush, K. A. Continuous functions without derivatives. Amer. Math. Monthly 59, 222–225 (1952).

The author gives an example of a class of continuous, non-differentiable functions $u=f(x)$ in $0\leqq x\leqq 1$. The construction is more elementary than the known examples, in the sense that no limit operations are involved beyond the representation of real numbers in the b-adic scale. If $x=.x_1x_2x_3\cdots$ is written in the b-adic scale ($b>2$), $u=.u_1u_2u_3\cdots$ is obtained in the dyadic scale ($u_k=0$ or 1), by defining $u_1=1$, and $u_{k+1}=$ or $\neq u_k$ according to whether $x_{k+1}=$ or $\neq x_k$. The continuity of $f(x)$ is easily verified. The non-differentiability is proved by constructing, for infinitely many p, a number $x^{(p)}$, differing from x in the pth digit only, such that $|f(x)-f(x^{(p)})|\geqq(2/3)2^{-p}$. As $|x-x^{(p)}|\leqq(b-1)b^{-p}$, it follows that

$$|(f(x)-f(x^{(p)}))/(x-x^{(p)})|\geqq\text{const.}\times(b/2)^p,$$

which shows that no finite differential coefficient exists. [Reviewer's note: it is easy to see that an infinite differential coefficient exists at a set of values dense in $0\leqq x\leqq 1$]. Cf. the following review. *F. A. Behrend* (Melbourne).

Wunderlich, W. Eine überall stetige und nirgends differenzierbare Funktion. Elemente der Math. 7, 73–79 (1952).

The author studies the properties of a continuous non-differentiable function due to Hilbert (lectures 1917) and published by H. Scherer [Semester-Berichte zur Pflege des Zusammenhangs von Universität und Schule (Münster) 12, 39–49 (1938)]. Except for the definition of u_1, the function is identical with that given by Bush [see the preceding review], for the case $b=3$. It is shown that the function satisfies the functional equations

$$\begin{aligned} f(x)&=\tfrac{1}{2}f(3x) && \text{for } 0\leqq x\leqq\tfrac{1}{3},\\ f(x)&=\tfrac{1}{2}f(x-\tfrac{1}{3}) && \text{for } \tfrac{1}{3}\leqq x\leqq\tfrac{1}{2},\\ f(x)&=f(1-x) && \text{for } \tfrac{1}{2}\leqq x\leqq\tfrac{2}{3},\\ f(x)&=1-f(1-x) && \text{for } \tfrac{2}{3}\leqq x\leqq 1, \end{aligned}$$

and that, together with the continuity of the function, these equations determine $f(x)$. *F. A. Behrend* (Melbourne).

Sierpiński, Wacław. Sur les suites doubles de fonctions. Fund. Math. 37, 55–62 (1950).

This paper is mainly concerned with interrelations of the following four properties for double sequences $\{f_{m,n}\}$ of real-valued functions on the real unit interval I. (A) Whenever $\{k_n\}$ and $\{l_n\}$ are sequences of natural numbers with $\lim_n k_n=\lim_n l_n=\infty$, then $\lim_n f_{k_n,l_n}(x)=0$ a.e. on I. (B) Same as (A) but with "a.e. on I" replaced by "except on a countable subset of I". (C) $\lim_{m,n} f_{m,n}(x)=0$ a.e. on I. (D) $\lim_{m,n} f_{m,n}(x)=0$ except on a countable subset of I.

Following are some of the statements proved. Every double sequence of measurable functions satisfying (A) also satisfies (C). If $2^{\aleph_0}=\aleph_1$, there exists a double sequence satisfying (B) but not (C). If $2^{\aleph_0}=\aleph_1$, there exists a double sequence of measurable functions satisfying (B) but not (D). Every double sequence of Baire functions satisfying (B) also satisfies (D). *T. A. Botts* (Charlottesville, Va.).

Arin'š, E. G. Some descriptive properties of monotone sequences of functions. Latvijas PSR Zinātņu Akad. Fiz. Mat. Inst. Raksti. 1, 101–104 (1950). (Russian. Latvian summary)

Let E be a complete separable metric space. A subset X of E is said to be residual if X' is of the first category in E. Let the real function f on E be the limit of an increasing sequence of upper semi-continuous functions. Then, on every perfect subset P of E, the points where f is lower-semi-continuous form a residual set. Let the property of Baire for a real function g on E be stated as follows: the points of continuity of g, considered on an arbitrary perfect subset P of E, form a residual set. Then the limit of a monotone sequence of functions enjoying the property of Baire also enjoys the property of Baire. The proofs are, of course, obvious. *E. Hewitt* (Seattle, Wash.).

Picone, Mauro. Su un criterio del Dini di convergenza uniforme. Boll. Un. Mat. Ital. (3) 7, 106–108 (1952).

Let $f, f_1, f_2, \cdots$ be functions on a closed compact set C in a Hausdorff space to a metric space. Let $\{f_n\}$ converge pointwise to f on C. Let each of the real-valued functions $|f_n-f|$ be upper semi-continuous on C. For this situation the author establishes a characterization of uniformity of convergence of $\{f_n\}$ to f which generalizes Dini's well-known theorem on uniform convergence of a monotone sequence of continuous real functions of a real variable. *T. A. Botts* (Charlottesville, Va.).

Cereteli, O. D. On an application of the theory of partially ordered spaces. Soobščeniya Akad. Nauk Gruzin. SSR. 12, 189–191 (1951). (Russian)

On a measurable set E [in an unspecified measure space] let there be given a sequence $\{x_n(t)\}_{n=1}^{\infty}$ of summable functions converging in measure to a function $x(t)$. The function $x(t)$ is summable and $\lim_{n\to\infty}\int_P x_n(t)dt=\int_P x(t)dt$ for every measurable $P\subset E$ if and only if every subsequence $\{x_{n_k}(t)\}_{k=1}^{\infty}$ contains a subsubsequence

$$\{x_{n_{k_i}}(t)\}_{i=1}^{\infty} \quad\text{with}\quad |x_{n_{k_i}}(t)|\leqq F(t) \quad (i=1, 2, \cdots),$$

where F is summable (F depends upon $\{n_k\}_{k=1}^{\infty}$). The proof is based on facts drawn from the theory of partially ordered linear spaces. *E. Hewitt* (Seattle, Wash.).

Volpato, Mario. Sulla derivazione sotto il segno di integrale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 146–150 (1952).

In an interval I: $c\leqq y\leqq d$, let $a(y)$, $b(y)$ be (real-valued) functions with the following properties. (i) $a(y)$, $b(y)$ are absolutely continuous and strictly monotone. (ii) $a(y)<b(y)$ for $c<y<d$. Denote by B the region $a(y)\leqq x\leqq b(y)$, $c\leqq y\leqq d$. In B, let there be given a function $f(x, y)$ satisfying the following conditions. (iii) $f(x, y)$ is continuous with respect to y and measurable with respect to x. (iv) There exists a summable function $M(x)$, such that $|f(x, y)|\leqq M(x)$, and $|f(x, y_2)-f(x, y_1)|\leqq M(x)|y_2-y_1|$ in B. For fixed y, denote by $F(y)$ the integral of $f(x, y)dx$ from $a(y)$ to $b(y)$. The purpose of the note is to show that $F(y)$ is absolutely continuous and $F'(y)$ can be calculated, almost everywhere, by the familiar formula relating to definite integrals with variable limits of integration. The proof is straightforward, except for the fact that the weak condition (iii) on $f(x, y)$ necessitates the use of a theorem of Dragoni concerning functions with the property (iii). *T. Radó*.

Kakutani, Shizuo. Quadratic diameter of a metric space and its application to a problem in analysis. Proc. Amer. Math. Soc. 3, 532–542 (1952).

In a metric space R with metric $d(x, y)$, the quadratic distance $d^{(2)}(x, y, S)$ of x and y with respect to a subset S of R is defined as inf $\sum_{i=1}^{p}[d(x_{i-1}, x_i)]^2$, taken for all possible finite chains $x=x_0, x_1, \cdots, x_p=y$ on S. The quadratic diameter of S, $\delta^{(2)}(S)=\sup_{x,y\in S} d^{(2)}(x, y, S)$. If any two points of S can be connected by a rectifiable curve, then $\delta^{(2)}(S)=0$. Examples of sets of positive quadratic diameter are given.

The following is proved: Let $f(u, v)$ be real, defined on an open square Q of 2-dimensional Euclidean space R^2 on which $\partial f/\partial u$ and $\partial f/\partial v$ exist and satisfy Lipschitz conditions. Then the set of values assumed by f at the set $S(f)$ of critical points of f in Q is of one-dimensional Lebesgue measure zero. Further, f is constant on every component of $S(f)$. The proof involves consideration of the areas of small circles with centers at the points of a chain, in the proof of the following lemma: If S is a compact subset of the closed unit square $\bar{Q}$: $\{0\leqq u, v\leqq 1\}$ of R^2 such that the two-dimensional Lebesgue measure of $(\bar{Q}-S)$ is less than ϵ, where $\epsilon<\pi/16$, then $\delta^{(2)}(S)<50\epsilon^{1/2}$.

The author states that the problem is open for the case of a function of more than 2 variables. However, as he states, if $f(x)$ is assumed to be of class $C^{(n)}$ in an open set of n-space, the conclusion of the theorem has been obtained by A. P. Morse [Ann. of Math. 40, 62–70 (1939)].

A. B. Brown (Flushing, N. Y.).

Kondrašov, V. I. The behavior of functions from L_p^ν on manifolds of different dimensions. Doklady Akad. Nauk SSSR (N.S.) 72, 1009–1012 (1950). (Russian)

This is a continuation of the author's investigation given in two earlier papers [C. R. (Doklady) Acad. Sci. URSS 48, 535–538 (1945); 51, 415–418 (1946); these Rev. 8, 32, 77] concerned with the space L_p^ν of functions of n variables which together with their partial derivatives of order at most ν have an L_p-norm. The functions are defined in a domain in a Euclidean space with a boundary $\sum_k S_{n-k}$, where each S_{n-k} is an $(n-k)$-dimensional manifold. If the manifold S'_{n-k} satisfies certain conditions of smoothness, the author asserts that a certain type of strong convergence on S'_{n-k} of a sequence of functions and their partial derivatives up to a certain order implies the convergence almost everywhere (with respect to measure on S'_{n-k}) of these functions and their partial derivatives. He also gives some results to the effect that the unit sphere in L_p^ν is compact in certain other spaces. *A. C. Offord* (London).

Burkill, J. C. Rearrangements of functions. J. London Math. Soc. 27, 393–401 (1952).

A result of Hardy and Littlewood concerning equimeasurable functions is extended from one to two variables. Let f be integrable in the rectangle R: $0\leqq x\leqq a$, $0\leqq y\leqq b$. Let $m(h)$ be the measure of the set for which $f(x, y)>h$. Let $C(h)$ be the curve of the family $\log(a/x)\log(b/y)=k$ which bounds an area $m(h)$ with the axes. Define f^* in R by requiring that $f^*(x, y)>h$ at all points between the axes and $C(h)$. Finally, let

$$\theta(f; x, y)=\sup \frac{1}{|S|}\int\int_S f(u, v)dudv \quad (0\leqq x'<x, 0\leqq y'<y),$$

where $|S|$ is the area of the rectangle S with southwest and northeast corners at (x', y') and (x, y). For $f\geqq 0$, the conclusion is

$$\int\int_R \theta(f; x, y)dxdy \leqq \int\int_R \theta(f^*; x, y)dxdy,$$

the latter integral being finite if $\int\int f\log^2(1+f)$ is finite. The proof is based on a variational argument. The appearance of Bessel functions is a rather unexpected feature of the analysis. *W. Rudin* (Rochester, N. Y.).

monotone

Silverman, Edward. Set functions associated with Lebesgue area. Pacific J. Math. 2, 243–250 (1952).

Let x be a continuous transformation from the square Q: $0\leqq u, v\leqq 1$ into m, the space of bounded sequences. The author has previously defined the Lebesgue area of the surface in m represented by x [Rivista Mat. Univ. Parma 2, 47–76 (1951); these Rev. 13, 122]. In the present paper he defines two set functions whose values on elementary configurations of m agree with the elementary areas of the configurations and whose values on the point set occupied by the surface represented by x are equal to the Lebesgue area of the surface. *R. G. Helsel* (Columbus, Ohio).

Federer, Herbert. Measure and area. Bull. Amer. Math. Soc. 58, 306–378 (1952).

Let X be any locally compact, locally connected, separable subset of a triangulable k-dimensional manifold and let $C_n(X)$ be the family of all continuous mappings $y=f(x)$, $x\in X$, $y\in E_n$, from X into E_n. Let $\mu(f, X, y)$, $y\in E_n$, be a function defined for each $f\in C_n(X)$ and suppose that, for each $y\in E_n$, $\mu(f, X, y)$ is either a non-negative integer, or ∞, and gives a sensible appraisal of the multiplicity with which the mapping f assumes the value y. Let Y_m, $m=0, 1, \cdots, \infty$, be the set of all $y\in E_n$ where $\mu(f, X, y)=m$, let $H_k(Y_m)$ be the k-dimensional Hausdorff measure of Y_m, and let $H_k(f)=H_k(Y_1)+2H_k(Y_2)+\cdots+(\infty)H_k(Y_\infty)$ (by $\infty\cdot a$ is meant 0 if $a=0$, ∞ if $a>0$). Then $H_k(f)$ can be considered as a k-dimensional area of the mapping f with respect to the multiplicity function μ. The Lebesgue area $L_2(f)$ of a mapping f from a 2-manifold X into E_n has been recently successfully studied and a great deal of information is now available, especially for the case $n=3$ and X a 2-cell. The main theorems for the Jordan length have been extended to the Lebesgue area, while many other definitions of area obtained by geometric, topological, integral-geometric considerations have been proved to coincide with the Lebesgue area. The Lebesgue area, because of its property of lower semicontinuity, is a powerful tool in the calculus of variations.

A k-dimensional Lebesgue area $L_k(f)$ can be defined for all mappings $f\in C_n(X)$, that is, under the same general conditions under which $H_k(f)$ has been defined above. Therefore the following important question arises: Is it possible to define the multiplicity function $\mu(f, X, y)$ in such a way that (*) $H_k(f)=L_k(f)$ for all $f\in C_n(X)$. In case X is a 2-cell and $n=3$ the present paper gives an affirmative answer to this question. For each subset $V\subset X$ let $\lambda_k(f, V)$ denote the k-dimensional Lebesgue area $L_k(f, V)$ of the mapping defined by f on V, if V is finitely triangulable. Otherwise let $\lambda_k(f, V)=\sup L_k(f, W)$ for all finitely triangulable subsets $W\subset V$. For any point $x\in X$ and $r>0$ let $\Delta_f(x, r)$ be the component containing x of the set of all $x'\in X$ such that $|f(x)-f(x')|<r$. Then each point $x\in X$ belongs to the subset A of X defined as the limit $A=\lim \Delta_f(x, r)$ as $r\to 0$. Any two sets A and B as above, and not disjoint, necessarily coincide, and $\Delta_f(x, r)$ is the same set for any $x\in A$ and is

denoted by $\Delta_f(A, r)$. The collection M_f of all the distinct sets A is a partition of X into disjoint sets. A metric can be introduced in M_f by defining as a distance $d_f(A, B)$ between two elements A, B of M_f the number $d_f(A, B) = \inf \operatorname{diam} f(W)$ for all compact connected subsets $W \subset X$ with $WA \neq 0$, $WB \neq 0$, whenever some sets W exist with such properties, otherwise $d_f(A, B) = \infty$. (The value ∞ for d_f can be avoided by replacing d_f by $d_f(1+d_f)^{-1}$.) M_f is the middle space of f and, as usual, by considering the monotone mapping m from X onto M_f which maps each point $x \in A$ into A itself, f has a monotone light factorization $f = lm$. Let $2(k)$ be the k-dimensional Lebesgue measure of the k-dimensional unit sphere in E_k and let $L_k^*(f, A)$, $L_{k*}(f, A)$ be the lim sup and lim inf as $r \to 0$ of the ratio $\lambda_k[f, \Delta_f(A, r)]/\alpha(k)r^k$ for each element $A \in M_f$. Then L_k^*, L_{k*} are the upper and lower densities of the Lebesgue area L_k at A on M_f. The author proves in various cases, for instance for $k=2$, $n=3$, X a 2-cell, that L_k^* and L_{k*} are integer and equal almost everywhere in M_f. This result is essentially the main tool for solving the problem stated above. The author makes use also of a characterization of Radó's multiplicity function for mappings from X into E_k in terms of Cech's cohomology theory, of various connections of the concepts above with Favard's integral-geometric measure, and of some previous results of the author [Trans. Amer. Math. Soc. 62, 114–192 (1947); these Rev. 9, 231] and of the reviewer [Ann. Scuola Norm. Super. Pisa (2) 10, 253–295 (1941); 11, 1–42 (1942); these Rev. 8, 257]. By defining the multiplicity function $\mu(f, X, y)$ as the sum $\mu(f, X, y) = \sum L_{k*}(f, A)$ for all $A \in M_f$ such that $l(A) = y$, one of the possible multiplicity functions μ for which (*) holds is obtained. *L. Cesari.*

Cesari, L., and Fullerton, R. E. On regular representations of surfaces. Rivista Mat. Univ. Parma 2, 279–288 (1951).

Using geometrical considerations alone, the authors prove that every non-degenerate Fréchet surface of finite Lebesgue area has a regular representation. This theorem is not new but previous proofs have relied upon the Dirichlet integral and other analytical devices. *R. G. Helsel.*

Theory of Functions of Complex Variables

***Fuks, B. A., i Levin, V. I. Funkcii kompleksnogo peremennogo i ih priloženiya. [Functions of a complex variable and their applications.]** Gosudarstv. Izdat. Tehn.-Teoret. Lit., Moscow-Leningrad, 1951. 307 pp. 11 rubles.

This book, subtitled "Special chapters," is intended to complement a book with a similar title by Fuks and Šabat [these Rev. 12, 87]. The chapters cover algebraic functions, differential equations, Laplace transforms, asymptotic expansions obtained by complex integration, and Hurwitz's problem on the zeros of polynomials. *R. P. Boas, Jr.*

Meier, Kurt. Zum Satz von Looman-Menchoff. Comment. Math. Helv. 25, 181–195 (1951).

In the first part of the paper the author gives an entirely elementary proof of the well-known theorem of Looman and Menchoff in the following somewhat simplified form: If (1) $f(z)$ $(z = x+iy)$ is continuous in a region G, (2) the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ exist at every point of G, and (3) the Cauchy-Riemann condition $\partial f/\partial x + i\partial f/\partial y = 0$ holds at every point of G, then $f(z)$ is holomorphic in G. As is known, in condition (2) an exceptional enumerable set of points and in condition (3) an exceptional set of two-dimensional Lebesgue measure zero may be admitted. The early part of the proof follows substantially Menchoff's method [cf. D. E. Menchoff, Les conditions de monogénéité, Hermann, Paris, 1936, pp. 9–16]. The basic tool used there is the special case of Baire's theorem which asserts that a perfect set in the plane is of the second category on itself. On the other hand, where Menchoff employs advanced real-variable technique, the author proceeds by means of an entirely elementary argument. The conclusion of the theorem follows from an elementary theorem of Lichtenstein [C. R. Acad. Sci. Paris 150, 1109–1110 (1910)]. The second part of the paper contains a proof of a result related to an earlier one of Menchoff's. In order to state this result, the following preliminary notions are introduced. Let z_0 be a point of a region G, let $z_0 + z(t)$ $(0 \leq t \leq 1)$, $z(0) = 0$, represent a Jordan arc C which emanates from the point z_0, and let $\varphi(t) = \arg z(t)$ be so chosen that $\varphi(t)$ is continuous in $0 < t \leq 1$. Let $\liminf_{t\to 0} \varphi(t) = \alpha$, $\limsup_{t\to 0} \varphi(t) = \beta$ and consider only such arcs C for which $\beta - \alpha < 2\pi$. Associate with C the "angular region" defined as the set of points $z_0 + \rho e^{i\psi}$ with $\rho > 0$, $\alpha \leq \psi \leq \beta$. A function $f(z)$ will be said to possess the property Q at the point z_0 if there exist three Jordan arcs C_j $(j = 1, 2, 3)$ which emanate from z_0 and whose associated angular regions are mutually disjoint such that the three limits

$$\lim_{z \to z_0,\, z \in C_j} \frac{f(z) - f(z_0)}{z - z_0} \quad (j = 1, 2, 3)$$

exist and are equal. The result in question may now be stated as follows: Let $f(z)$ be continuous in a region G and let $f(z)$ possess the property Q at all points of G with the possible exception of an enumerable set of points. Then $f(z)$ is holomorphic in G. *W. Seidel* (Princeton, N. J.).

Waadeland, Haakon. On some transcendental equations. I. Norske Vid. Selsk. Forh., Trondheim 24 (1951), 16–19 (1952).

If z is a root of the transcendental equations

$$\cosh z \cos z = \pm 1,$$

then $-z$, $\pm iz$, $\pm \bar{z}$, and $\pm i\bar{z}$ are also roots. The purpose of this paper is the proof that there are no roots lying off the axes of the Gaussian plane. *E. Frank* (Chicago, Ill.).

Svirskiĭ, I. V. The determination of the number of roots lying in the right half-plane for functions of the form $F(e^z, z)$, where $F(e^z, z)$ is a rational function of the arguments e^z and z and an application of the results to the investigation of automatic regulation of steam turbines. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk 1, 51–61 (1948). (Russian)

The rational function $F(e^z, z)$ is assumed to be linear in e^z. A rotation and a substitution reduce the problem to that of finding the zeros in the lower half-plane of $F(z) = \cot(z/2) - \Phi(z)$, where $\Phi(z)$ is a rational function. The author then gives a detailed analysis of $\Delta \arg F(z)$ on a suitable contour, but the formula obtained seems to be in error due to confusion about the positive direction on the contour. In the application to the steam turbine, the last assertion of the paper is unjustified because of the unknown nature of the physical constants involved.

A. W. Goodman (Lexington, Ky.).

Tôyama, Hiraku. On some determinant equation. Kōdai Math. Sem. Rep. 1952, 31–32 (1952).

Let $A(z)$ be a square matrix of order n such that (i) every element $A_{ik}(z)$ of $A(z)$ is an entire function of z,

$$A_{ik}(z)=\sum_{m=0}^{\infty}A_{ik,m}z^m,\quad A_{ik}(0)=0,$$

(ii) every coefficient $A_{ik,m}\geqq 0$, (iii) at least one coefficient of the characteristic polynomial $|\lambda E-A(z)|$ is not constant. Then the determinant equation $|E-A(z)|=0$ has a real positive root of smallest absolute value. *E. Frank.*

Sz.-Nagy, Gyula. Über die Lage der kritischen Punkte rationaler Funktionen. Acta Sci. Math. Szeged 14, 179–185 (1952).

Given the nth degree rational function $R(z)=f(z)/g(z)$, where $f(z)=a_0+a_1z+\cdots+a_nz^n$ and $g(z)=b_0+b_1z+\cdots+b_nz^n$ with $|a_n|+|b_n|\neq 0$, and with $f(z)$ and $g(z)$ possessing no common factor. The index m of $R(z)$ is defined as the maximum k for which $a_kb_n\neq b_ka_n$. The Z-points of $R(z)$ mean the zeros of the polynomial $[f(z)-Zg(z)]$, and the critical points of $R(z)$ mean the zeros of $D(z)=f'(z)g(z)-g'(z)f(z)$, of which there are $n+m-1$ counting multiplicities. For $\alpha\delta-\beta\gamma\neq 0$, the rational function

$$S(z)=[\alpha R(z)+\beta]/[\gamma R(z)+\delta]$$

has the same degree, index, and critical points as R and the Z-points of R are the W-points of S with

$$W=(\alpha Z+\beta)/(\gamma Z+\delta).$$

In particular, the choice $S=(R-A)/(R-B)$ shows that known theorems on the location of the critical points of R relative to the zeros and poles of R may be restated as theorems on the location of the critical points of R relative to the A and B points of R. [Reviewer's note: Similar results for $R(z)\equiv f(z)$ are given in Marden, The geometry of the zeros ..., Amer. Math. Soc., New York, 1949, pp. 22–23; these Rev. 11, 101]. *M. Marden* (Milwaukee, Wis.).

Dalovitch, Voïn. Sur l'existence des valeurs limites sur le bord du cercle-unité de la résultante de deux fonctions. Bull. Soc. Math. Phys. Serbie 3, nos. 3–4, 51–55 (1951). (Serbo-Croatian. French summary)

Dans cet article on demontre le théorème suivant: Si la fonction $f(z)=\sum_{n=0}^{\infty}a_nz^n$ appartient à la classe de fonctions H_2, elle possède une majorante $g(z)=\sum_{n=0}^{\infty}b_nz^n$ telle que la résultante $F(z)=\sum_{n=0}^{\infty}a_nb_nz^n$ de ces deux fonctions soit fonction entière de $1/(1-z)$ et qu'elle ait par conséquent une valeur limite dans tous les points sur le bord du cercle-unité, excepté dans $z=1$ au plus. (Author's summary.)
W. H. J. Fuchs (Ithaca, N. Y.).

Dahlquist, Germund. On the analytic continuation of Eulerian products. Ark. Mat. 1, 533–554 (1952).

Let $h(z)$ be analytic in $|z|\leqq 1$, except for isolated singularities, and $h(0)=1$. Construct the "Euler product" $f(s)=\prod_p h(p^{-s})$, where p runs over the primes. The author first shows that $f(s)$ is analytic in $\sigma>0$ $(s=\sigma+i\tau)$ except on an isolated point set. His main result is Theorem I: The imaginary axis is a natural boundary of $f(s)$, except when $h(z)$ is of the form $\prod_{\nu=1}^{k}(1-z^\nu)^{-\beta_\nu}$; in the exceptional case $f(s)=\prod_{\nu=1}^{k}\zeta(\nu s)^{\beta_\nu}$ and is analytic in the whole plane except for isolated points. The author makes the proof by uniquely factorizing $h(z)$: $h(z)=\prod_{\nu=1}^{\infty}(1-z^\nu)^{-\beta_\nu}$, $|z|<a$, which leads easily to the representation (not unique)

$$f(s)=\prod_{p\leqq q}h(p^{-s})\prod_{\nu=1}^{\infty}\zeta_q(\nu s)^{\beta_\nu},\quad\text{where}\quad \zeta_q(s)=\zeta(s)\prod_{p\leqq q}(1-p^{-s}),$$

q satisfies $q>a^{-1/\sigma}$, and a is the smaller of the numbers 1 and the modulus of the smallest zero or singularity of $h(z)$. Thus he can use known results on the distribution of the zeros of $\zeta(s)$. The proof depends crucially on Lemma 3.2, a general theorem on sequences of positive integers. In the last part of the paper the author extends his results to the class

$$f(s)=\prod_p h(\chi(p)p^{-s}),$$

h satisfying the same conditions as before, and χ a group character to a fixed modulus. *J. Lehner.*

Vekua, N. P. Generalized Hilbert boundary problem for several unknown functions. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 577–584 (1947). (Russian)

Let D^+ be the bounded domain, whose frontier is a simple, closed, smooth contour L (origin in D^+); D^- = complement of D^++L. Let $\alpha(t)$, εH (Hölder on L), with $\alpha^{(1)}(t)\neq 0$ on L, be a function transforming L one-to-one on itself (t, $\alpha(t)$ describing L in the same sense); $\beta(t)$ denotes the inverse of $\alpha(t)$. The generalized Hilbert problem consists in finding piecewise analytic functions $\phi_j(z)$, having finite order at ∞, so that (1) $\phi^+[\alpha(t_0)]=G(t_0)\phi^-(t_0)+g(t_0)$ on L, where $\phi=(\phi_1,\cdots,\phi_n)$, $g=(g_1,\cdots,g_n)$ are vectors and $G(t_0)=(G_{i,j})$ $(i,j=1,\cdots,n)$ is a matrix; the $G_{i,j}$, g_k are assigned on L and are in H; $\det(G_{i,j})\neq 0$ (on L). The original Hilbert problem is on hand if $\alpha(t)\equiv t$. First is solved the homogeneous problem (2) $\phi^+[\alpha(t_0)]=G(t_0)\phi^-(t_0)$; (2) is equivalent to a pair of integral equations in the sense of principal values. Every solution of (2) of order at infinity not exceeding r, where r is an assigned suitably great integer, is of the form (2′) $\phi(z)=\sum_1^m\gamma_j\overset{j}{\phi}(z)$, where the γ_j are arbitrary constants, the $\overset{j}{\phi}$ are certain distinct particular solutions, of which the first n satisfy $\lim_{z\to\infty}z^{-r}\overset{k}{\phi}(z)=\overset{k}{\gamma}$ $(k=1,\cdots,n)$, the $\overset{j}{\phi}$ $(j>n)$ having at ∞ order less than r. The result (2′) leads to a canonical solution of (2), $X=(\overset{j}{\chi}_i)$ $(i,j=1,\cdots,n)$; here $\det X(z)$ is non-zero for finite z; $\Delta^0(z)=\det(z^{-H_j}\overset{j}{\chi}_i(z))$ is finite and is distinct from zero at infinity, the H_j being certain numbers independent of the choice of X. All solutions of (2) are given by

$$(3)\quad 2\pi i\phi(z)=\begin{cases}X(z)\int\rho[\beta(t)](t-z)^{-1}dt & (\text{in } D^+),\\ X(z)\int\rho(t)(t-z)^{-1}dt+X(z)p(z) & (\text{in } D^-),\end{cases}$$

where $p(z)$ is a polynomial vector and $\rho(t)$ is the solution of a certain regular non-homogeneous Fredholm equation $T\rho=p$. The non-homogeneous problem (1) is solved by (3), where ρ is a solution of $T\rho=\{X^+[\alpha(t_0)]\}^{-1}g(t_0)+p(t_0)$. A detailed exposition of the above is to be published elsewhere.
W. J. Trjitzinsky (Urbana, Ill.).

Vekua, N. P. The generalized Hilbert boundary problem for several unknown functions. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 16, 81–103 (1948). (Georgian. Russian summary)

"The basic results have been announced in the Soobščeniya Akad. Nauk Gruzin. SSR 8, 577–584 (1947)" [see the paper reviewed above].
From the author's summary.

Magnaradze, Leo. On a generalization of the theorem of Plemelj-Privalov. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 509–516 (1947). (Russian)

Let L be a simple, closed, rectifiable Jordan curve: $t=x(s)+iy(s)$ $(0\leqq s\leqq l=\text{length of } S)$; the integral

$$(1)\qquad g(t_0)=\frac{1}{2\pi i}\int_L \frac{f(t)}{t-t_0}dt$$

has a meaning in the sense of Cauchy principal values, provided $f(t)$ is of an appropriate class (K). The author considers the problem of finding a subclass (K_0) of (K) so that, when f is in (K_0), so is g. This is achieved by introduction of certain remarkable classes of continuous functions. Let $\omega(\tau;f)=\sup(|s_2-s_1|\leqq\tau)|f(t_2)-f(t_1)|$ $(0<\tau\leqq l)$, $I(\tau;f)=\tau^{-1}\omega(\tau;f)$; the class I_p is that of functions f such that $\int_0^l I(\tau;f)(\log(l/\tau))^p dt<+\infty$ $(p\geqq 0)$. The functions of I_p are continuous; I_p contains the class (H) (Hölder); every f of I_p satisfies a generalized Dini condition: $\lim_{\tau\to 0}\omega(\tau,f)(\log(l/\tau))^{p+1}=0$. If $f\in I_0$, then the principal integral (1) exists and g is continuous. $I_q\geqq I_p$ for $0\leqq q<p$. The class $I_\infty=\prod_{p\geqq 0}I_p$ contains (H) and I_∞ contains certain functions not in (H). The author generalizes the Plemelj-Privaloff theorem by proving that if $f\in I_\infty$, then $g\in I_\infty$. Let L be a simple, closed, smooth curve; consider

$$(2)\qquad \Phi(z)=\frac{1}{2\pi i}\int_L\frac{\phi(t)dt}{t-z}.$$

It is shown that if $\phi\in I_\infty$, then $\Phi\in I_\infty$. These considerations are intended to lead to eventual generalizations of the existing theories of Riemann-Hilbert boundary value problems (for analytic functions) and of integral equations in the sense of principal values. *W. J. Trjitzinsky* (Urbana, Ill.).

Magnaradze, Leo. On a linear boundary problem of Riemann-Hilbert. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 585–590 (1947). (Russian)

The problem considered is to find $\phi^+(z)$, analytic in S^+, and $\phi^-(z)$, analytic in S^- (S^+ and S^- being the interior and exterior domains with respect to a simple, closed, smooth contour L) and of finite order at infinity, so that (1) $\phi^+(t)=a(t)\phi^-(t)+b(t)$ on L, $a(t)$ and $b(t)$ being assigned on L. Use is made of an earlier work of the author [see the preceding review]. The author constructs an effective solution of (1), when $a(t)$, $b(t)$ are of class I_∞ and the boundary values $\phi^+(t)$, $\phi^-(t)$ are attained within S^+, S^-, respectively, along any paths. The validity of Plemelj formulas for the integral

$$\phi(z)=\frac{1}{2\pi i}\int_L\frac{\mu(t)}{t-z}dt$$

is established when $\mu(t)\in I_0$, in which case the limits $\phi^+(t)$, $\phi^-(t)$ are attained for t on L, as stated above; furthermore, when $\mu(t)$ is in I_∞, the boundary values ϕ^+, ϕ^- are also in I_∞. The general solution of (1) is then obtained in the form

$$\Phi(z)P(z)+\frac{\Gamma 1}{2\pi i}\Phi(z)\int\frac{b(t)dt}{\Phi^+(t)(t-z)},$$

with $\Phi(z)=e^{H(z)}$ (in S^+), $=z^{-n}e^{H(z)}$ (in S^-), and $2\pi iH(z)=\int\log[t^{-n}a(t)](t-z)^{-1}dt$, $n=(2\pi i)^{-1}\int d\log a(t)$,

$P(z)$ = an arbitrary polynomial. The above considerations can be applied to develop a corresponding theory of integral equations in the sense of principal values.

W. J. Trjitzinsky (Urbana, Ill.).

Magnaradze, Leo. On the tangent derivative of the logarithmic potential of a simple layer. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 591–596 (1947). (Russian)

Classes I_p $(0\leqq p\leqq+\infty)$ have been previously defined and studied by the author [see the second review above]. Let L be a simple, closed, rectifiable Jordan curve, $x=x(s)$, $y=y(s)$, $0\leqq s\leqq l$ (l = length of L); the derivatives $x'(s)$, $y'(s)$ are assumed to be in I_0, $(x')^2+(y')^2>0$ on L. The potential $U(x,y)=(1/2\pi)\int_L\mu(\sigma)\log r d\sigma$ ($d\sigma$ = element of arc), where μ is of period l and is in I_0, is continuous in every finite part of the plane; on L one has

$$U(s)\equiv U(x(s),y(s))=\frac{1}{2\pi}\int_0^l\mu(\sigma)\log r(s,\sigma)d\sigma$$

and

$$U'(s)=\text{Pr.V.}\ \frac{1}{2\pi}\int_0^l\mu(\sigma)\frac{\partial\log r(s,\sigma)}{\partial s}d\sigma$$

exists. *W. J. Trjitzinsky* (Urbana, Ill.).

Karcivadze, I. N., and Hvedelidze, B. V. On a transformation formula. Soobščeniya Akad. Nauk Gruzin. SSR. 10, 587–591 (1949). (Russian)

Let C_k $(k=1,2,\cdots)$ be simple, closed, smooth, with a positive direction assigned; $C=\sum_1^\infty C_k$; C_k and C_j $(k\neq j)$ exterior to each other. Classes I_p, I_∞ have been defined and studied by L. G. Magnaradze [see the three preceding reviews]. φ belongs to A on C, if φ is in I_∞ on the C_k and

$$(1)\qquad v(t)=\sum\int_{C_k}\varphi(\zeta)\frac{d\zeta}{\zeta-t}$$

converges uniformly on every C_k, $v(t)$ being in I_∞ on the C_k. It is proved that, if φ is in A, then

$$\varphi(x)=\frac{1}{\pi i}\int_C\frac{dt}{t-x}\left\{\frac{1}{\pi i}\int_C\frac{\varphi(\zeta)d\zeta}{\zeta-t}\right\}\quad(x,t,\zeta\text{ on }C).$$

As a consequence, the integral equation

$$(2)\qquad a\varphi(t)+\frac{b}{\pi i}\int_C\frac{\varphi(\zeta)d\zeta}{\zeta-t}=f(t)$$

[t on C, f in A on C, a, b constants, $a^2-b^2\neq 0$] has a unique solution in A. $\varphi\in A_p[A_p^*]$, if $\varphi\in I_p[I_{p-1}]$ on every C_k, series (1) converges unformly on every C_k, $v(t)\in I_{p-1}[I_p]$ on each C_k. If f is in A_p, the equation (2) has a unique solution in $A_p^*(p\geqq 2)$. *W. J. Trjitzinsky* (Urbana, Ill.).

Kveselava, D. A. Some boundary problems of the theory of functions. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 16, 39–80 (1948). (Russian. Georgian summary)

The author uses largely the notation and terminology of N. I. Mushelišvili [Singular integral equations . . . , Moscow-Leningrad, 1946; these Rev. 8, 586]. One of the problems considered is that of finding a piecewise analytic function $\Phi(z)$, such that (1) $\Phi^+[\alpha(t)]=G(t)\Phi^-(t)+g(t)$ on L, $G(t)\neq 0$ on L, where $G(t)$, $g(t)$ are of class H on L; here $\alpha(t)$ transforms L one-to-one into itself, t and $\alpha(t)$ describing L in the same direction. First the author solves explicitly the problem [(1), $G(t)=1$]; this involves contour integrals with the Cauchy kernel. Next he solves the homogeneous problem [(1), $g(t)=0$]; this involves use of the canonic function corresponding to $G(t)$ and of the notion of index H. These results lead to the explicit formulas expressing every piecewise analytic solution of the non-homogeneous problem (1); the formulas involve an arbitrary constant, two arbitrary

rational functions, the canonic function and a solution of a singular integral equation. When the index is negative, the problem can be solved if and only if g is orthogonal on C' to the g_k ($k=0, \cdots, -H-1$), where the $g_k(t)$ constitute a certain set of linearly independent functions. Also examined is the case when t and $\alpha(t)$ describe L in opposite directions, the whole equation (1) is replaced by

$$(2) \qquad \Phi^+[\alpha(t)] = G(t)\Phi^-(t) + g(t).$$

In this connection the sign of the index is important; for lack of space we cannot give detailed statements of the results. Also solved is the problem (generalizing a problem of Carleman) of finding $\Phi(z)$, meromorphic in S^+ such that (3) $\Phi^+[\alpha(t)] = G(t)\Phi^+(t) + g(t)$ on L, with G and g in H, $G \neq 0$ on L, t, $\alpha(t)$ describing L in opposite directions, while $\alpha[\alpha(t)] = t$. This problem is solved explicitly along the lines used in solving (1), (2). Every solution of (3) is of the form

$$\Phi(z) = C + R(z) + \frac{1}{2\pi i}\int_L \frac{\phi(\tau)d\tau}{\tau - z},$$

where C is a constant, $R(z)$ is an arbitrary principal part of $\Phi(z)$ and $\phi(t)$ is a solution of a certain quasi-regular Fredholm equation. The general solution is further studied with great completeness and detail with the aid of the canonic function and the index. *W. J. Trjitzinsky* (Urbana, Ill.).

Vekua, N. P. On a boundary problem of the theory of functions of a complex variable for several unknown functions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 157–180 (1952). (Russian)

Let L be a closed, "smooth" contour, bounding a finite domain D^+ in the plane U of $z = x + iy$; $D^- = U - D^+ - L$; the origin of coordinates in D^+; it is assumed that the angle between the tangent to L and a fixed direction is of class H (Hölder); $\alpha(t)$ is assigned on L; $\alpha^{(1)}(t)$, $\neq 0$ on L, belongs to H; $\alpha(t)$ effects a one-to-one transformation of L into itself, reversing the direction along L; the inverse of $\alpha(t)$ is denoted by $\beta(t)$. It is said that $\phi(z)$ is meromorphic in D^+ (in D^-) if $\phi(z)$ is analytic in D^+ (in D^-), except for a finite number of possible poles, and if $\phi(z)$ is continuously extensible to L. The author considers the problem of finding two vectors $\phi_i = [\phi_{i1}, \cdots, \phi_{in}]$ ($i = 1, 2$) meromorphic in D^+, so that (1) $\phi_1^+[\alpha(t_0)] = G(t_0)\phi_2^+(t_0) + g(t_0)$ (on L), with the matrix $G(t_0) = (G_{kj})$ ($k, j = 1, \cdots, n$) and the vector $g = [g_1, \cdots, g_n]$ both assigned and of class H; det $G(t) \neq 0$ on L. If it is required that $\phi_1 = \phi_2$, problem (2) of T. Carleman [Verh. Internat. Math.-Kongresses, Zürich, Bd. I, Füssli, Zürich-Leipzig, 1932, pp. 138–151] is at hand. The author utilizes some results by Kveselava [see the preceding review], referring to the case $n = 1$, and solves the general problem (1) with the aid of a generalization of a method due to Plemelj [Monatsh. Math. Phys. 19, 211–246 (1908)]. The problem (2) involves the use of integral equations in the sense of principal values; the latter problem is also treated in the case of discontinuous coefficients.

W. J. Trjitzinsky (Urbana, Ill.).

Krikunov, Yu. M. On the solution of a generalized boundary problem of Riemann and of a linear singular integro-differential equation. Doklady Akad. Nauk SSSR (N.S.) 85, 269–272 (1952). (Russian)

Let $\phi^+(z)$, $\phi^-(z)$ be, respectively, analytic interior and exterior to a simple, closed, smooth contour L (origin in L); $\phi^-(\infty) = 0$. It is shown that, if the derivatives $\phi^{+(m)}(z)$, $\phi^{-(n)}(z)$ exist on L and are in H (Hölder class on L), then

$$\varphi^+(z) = \frac{(-1)^m}{(m-1)!}\frac{1}{2\pi i}\int_L \nu d\tau + \sum_{k=0}^{m-1}\frac{c_k z^k}{k!},$$

$$\varphi^-(z) = \frac{(-1)^n}{(n-1)!}\frac{1}{2\pi i}\int_L \sigma d\tau,$$

where $\nu = \mu(\tau)(\tau - z)^{m-1}\ln(1 - z/\tau)$,

$$\sigma = (\mu(\tau)/\tau^n)[(\tau - z)^{n-1}\ln(1 - \tau/z) + \sum_0^{n-2}\beta_k \tau^{n-k-1} z^k],$$

$\mu(\tau)$ is a certain function defined on L, the c_k are arbitrary constants, $\ln(1 - z/\tau) = 0$ for $z = 0$, $\ln(1 - \tau/z) = 0$ for $z = \infty$, the β_k are certain constants. The above result enables one to reduce to an integral equation (in the sense of principal values and of familiar type) the following Riemann problem. To find $\phi^+(z)$, $\phi^-(z)$, if on L one has

$$\sum_{k=0}^m \left[a_k(t)\phi^{+(k)}(t) + \int_L A_k(t, t_1)\phi^{+(k)}(t_1)dt_1\right] - \sum_{k=0}^n \left[b_k(t)\phi^{-(k)}(t) + \int_L B_k(t, t_1)\phi^{-(k)}(t_1)dt_1\right] = f(t);$$

here a_k, b_k, f are in H; a_m, b_n are non-zero on L;

$$A_k = |t_1 - t|^{-\lambda}A_k^0(t, t_1), \quad B_k = |t_1 - t|^{-\lambda}B_k^0(t, t_1) \quad (0 \leqq \lambda < 1),$$

A_k^0, B_k^0 being in H. On the other hand, there is an integro-differential equation (in the sense of principal values), whose solution is reduced to that of the above boundary problem. *W. J. Trjitzinsky* (Urbana, Ill.).

Block, I. Edward. The Plemelj theory for the class Λ^* of functions. Duke Math. J. 19, 367–378 (1952).

Soit $f(z) = F(s)$ une fonction définie et sommable sur une courbe fermée simple rectifiable Γ, frontière d'un domaine B (s = arc de la courbe); on considère les fonctions

$$\alpha_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta \quad (z \text{ intérieur à } \Gamma);$$

$$\alpha_2(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta \quad (z \text{ extérieur à } \Gamma).$$

On pose les définitions suivantes: (i) $f(z)$ appartient à la class Λ^* (Zygmund) si $|F(s+h) - 2F(s) + F(s-h)| \leqq K|h|$ (K constante indépendante de s et de h); (ii) $f(z)$ appartient à la classe Λ_n^* si sa dérivée $n$ème supposée existante appartient à Λ^*; (iii) $\alpha_1(z)$ appartient à Z_A^n si elle est continue sur $B + \Gamma$ et si sa valeur sur Γ appartient à Λ_n^* (on a une définition analogue pour α_2); (iv) la courbe Γ est dite du type G (Walsh et Elliot) si une fonction $z = \psi(w)$, qui fait la représentation conforme de la région C: $|w| < 1$ sur B, est telle que $\psi'(w)$ est continue sur $C + \gamma$ (γ frontière de C), différente de zéro sur γ, et satisfait à une condition de Lipschitz (d'ordre positif) sur γ. Alors, si $f(z) \in \Gamma_n^*$ et si Γ est du type G, α_1 et α_2 appartiennent à Z_A^n sur Γ, et leurs dérivées $n$èmes ont des limites sur Γ qui sont les dérivées $n$èmes de f_1 et f_2. Ce résultat généralise des résultats antérieurs de Plemelj, Privaloff et Davydov. *J. Favard.*

Cowling, V. F. On functions defined by Taylor and Newton series. Univ. Nac. Tucumán. Revista A. 8, 41–47 (1951).

The possibility of representing an analytic function $f(z) = \sum a_n z^n$ by an integral

$$\int_0^1 \frac{\phi(x)dx}{1 - xz}$$

taken along the real axis was first studied by Le Roy. Here in the case in which there exists a function $a(t)$ analytic in the half-plane $x \geqq \rho$ $(0<\rho<1)$, $a(t)=o(t^{-h})$ for some $h>1$, such that $a(n)=a_n$ $(n=1, 2, \cdots)$, the solution of the problem is presented in closed form:

$$f(z)=\frac{z}{2\pi i}\int_0^\infty \frac{du}{e^u-z}\int_{\rho-i\infty}^{\rho+i\infty} e^{ut}a(t)dt.$$

An analogous result is obtained for series of Newton.
E. N. Nilson (Hartford, Conn.).

Kober, H. A remark on the approximation of a function of two real variables by nearly analytic functions. J. London Math. Soc. 27, 369–371 (1952).

Une fonction de la variable $z=x+iy$ est dite presque analytique dans une région si elle y est continue et si elle est analytique en tout point, sauf, peut-être, sur un ensemble de mesure superficielle nulle. Soit alors $\phi(x, y)$ une fonction continue pour $x^2+y^2 \leqq 1$; il existe une suite $\{h_n(z)\}$ de fonctions presque analytiques, uniformément convergentes dans cette région vers ϕ, et telle que $h_n'(z)=g'(z)$ presque partout, $g(z)$ désignant une fonction analytique à dérivée bornée pour $|z|<1$; si ϕ satisfait à une condition de Lipschitz, on peut s'arranger pour qu'il en soit de même pour les h_n dans $x^2+y^2 \leqq 1$. La démonstration utilise une fonction des sauts, déjà etudiée par l'auteur [Trans. Amer. Math. Soc. **67**, 433–450 (1949); ces Rev. 11, 336]; l'intérêt du résultat est limité par sa généralité même. *J. Favard* (Paris).

Tumarkin, G. C. On approximation in the mean of complex-valued functions. Doklady Akad. Nauk SSSR (N.S.) **84**, 21–24 (1952). (Russian)

Let $L^p(d\sigma)$ denote the space of functions $f(t)$ for which $\int_0^{2\pi}|f(t)|^p d\sigma(t)$ is finite. It was shown by Ahiezer [Lectures on the theory of approximation, Moscow-Leningrad, 1947, p. 278; these Rev. **10**, 33] that for $p \geqq 1$ the set $\{e^{int}\}_{n=0}^\infty$ is closed in $L^p(d\sigma)$ if and only if (*) $\int_0^{2\pi}\log \sigma'(t)dt=-\infty$; the case $p=2$ is ascribed to Kolmogoroff [Byull. Moskov. Gos. Univ. Mat. 2, no. 6 (1941); these Rev. **5**, 101] and Krein [C. R. (Doklady) Acad. Sci. URSS **46**, 91–94 (1945); these Rev. 7, 156]. The author extends this to the case $p>0$, discusses the span of $\{e^{int}\}$ when (*) is not satisfied, and considers some related questions. If (*) does not hold, approximation to $f(t)$ of $L^p(d\sigma)$ by a polynomial in e^{it} is possible if and only if $F(e^{it})=f(t)$ coincides almost everywhere on $|z|=1$ with the boundary values of a function $F(z)$ having $\int_0^{2\pi}\log^+|F(re^{it})|dt$ bounded. Extensions and applications deal with the case where the unit circle is replaced by a Jordan region. Corresponding results are given for functions belonging to $L^p(d\sigma)$ on the whole real axis [cf. Ahiezer, loc. cit., p. 280, for the case where the analogue of (*) holds]. *R. P. Boas, Jr.* (Evanston, Ill.).

Haplanov, M. G. A matrix criterion of completeness of a system of analytic functions. Doklady Akad. Nauk SSSR (N.S.) **83**, 35–38 (1952). (Russian)

Using methods from an earlier paper [same Doklady **79**, 929–932 (1951); these Rev. **13**, 252] the author generalizes a result of Markuševič [Mat. Sbornik **17(59)**, 211–252 (1945); these Rev. **7**, 425]. Let M be a matrix transforming the analytic space E into the analytic space E_1. The principal result is: in order that the system of elements Mx^1, Mx^2, $\cdots$ be complete (i.e., fundamental) in $ME \subset E_1$, it is necessary and sufficient that the functionals ux^1, ux^2, $\cdots$, with $u \in E^*$, not all vanish at any $u_0 \neq 0$ in $M'E_1^*$ (M' = transpose of M). If $E=\bar{A}_{1/\rho}$, this result reduces to Markuševič's "Duality Principle". *B. Crabtree* (Durham, N. H.).

Röhrl, Helmut. Zur Theorie der Faberschen Entwicklungen auf geschlossenen Riemannschen Flächen. Arch. Math. **3**, 93–102 (1952).

The author extends the results of Tietz [J. Reine Angew. Math. **190**, 22–33 (1952); these Rev. **13**, 833] to obtain series expansions of regular analytic functions or differentials in terms of elementary functions or differentials, resp., which are characterized by more general divisors than those used by Tietz. *G. Springer* (Evanston, Ill.).

Leont'ev, A. F. Generalization of Liouville's theorem. Mat. Sbornik N.S. **31(73)**, 201–208 (1952). (Russian)

Liouville's theorem, after a change of variable, may be stated as follows: an entire function $F(z)$ of the form $\sum_0^\infty A_n e^{nz}$, such that $|F(z)| \leqq |e^z|^\mu$ for some μ, is necessarily a polynomial in e^z. The author generalizes this statement as follows. Let $f(z)=\sum_0^\infty a_n z^n$, $a_n \neq 0$, be an entire function of order ρ and positive type, with $\liminf n|a_n|^{\rho/n}>0$. Let λ_n be distinct complex numbers, $|\lambda_n| \uparrow \infty$, and

$$\limsup n|\lambda_n|^{-\rho}<\infty.$$

Then if the entire function $F(z)$ is the limit, uniformly in every bounded region, of sums $\sum_{j=1}^{p_n} a_{n,j} f(\lambda_j z)$, and is at most of finite type of order ρ, then it is a finite linear combination of functions $f(\lambda_n z)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Littlewood, J. E. On some conjectural inequalities, with applications to the theory of integral functions. J. London Math. Soc. **27**, 387–393 (1952).

Let $g(z)$ be a rational function of degree at most N, $h(g)=|g'|/(1+|g|^2)$, $J(r)=r^{-1}\int_0^r\int_{-\pi}^{\pi} h(g)rdrd\theta$. The main conjectural inequality is that $\psi(N)$, the upper bound of $J(r)$ for varying g and r, does not exceed $AN^{\frac{1}{2}-\alpha}$ with positive absolute constants A, α. That $\psi(N) \leqq 2\pi N^{\frac{1}{2}}$ is easy, and a counterexample shows that $\psi(N)<A$ is false. The inequality implies the following result on the distribution of the values of an entire function $f(z)$ of finite positive order ρ. Let $0<\beta<\alpha$. There is some w_0 such that for almost every w all but $O(r^{\rho-(\alpha-\beta)\rho+\epsilon})$ of the w-points of $f(z)$ in $|z| \leqq r$ lie in a set of circles having centers at points z where $f(z)=w_0$, $|z| \leqq r$, and radii $9|z|^{1-\frac{1}{2}\rho-\beta\rho}$; or, as the author describes the situation informally, there is an infinitesimal portion S of the z-plane such that for almost every w the w-points of $f(z)$ lie, with negligible exceptions, in S. *R. P. Boas, Jr.*

Videnskiĭ, V. S. Consequence of a proposition of S. N. Bernšteĭn on entire functions of genus zero. Doklady Akad. Nauk SSSR (N.S.) **84**, 421–424 (1952). (Russian)

The proposition in question is Theorem 3 of the same Doklady (N.S.) **77**, 549–552 (1951); these Rev. **12**, 814. The author proves the following theorems. (1) If $e^{p(x)}$ is in class N [definition in the cited review] and $\int^\infty x^{-2}p(x)dx<\infty$, then there is an even entire function $F(x)$ of genus 0 with nonnegative Maclaurin coefficients such that $e^{p(x)}<F(x)$ on the real axis. As a consequence we have a new proof of a theorem of Bernšteĭn on weighted polynomial approximation [ibid. **77**, 773–776 (1951); these Rev. **13**, 26]. (2) The even entire function $f(z)=\sum c_{2n}z^{2n}$ is of genus 0 if and only if $\psi(z)=\sum|c_{2n}|z^{2n}$ is of genus 0. The proof depends on another theorem of Bernšteĭn [ibid. **66**, 545–548 (1949); these Rev. **11**, 23]. As corollaries, we have that $f(z)$ is of genus 0 if and only if $\int^\infty x^{-2}\log|\psi(x)|dx$ converges, or if and only if $\sum c^*{}_{2n}$ and $\sum \bar{c}_{2n}{}^{1/(2n)}$ converge, where

$$c^*{}_{2n}=\max_{k\geq 0}|c_{2(n+k)}|^{1/(2n+2k)}$$

and $\{\tilde{c}_{2n}\}$ is Valiron's rectified sequence corresponding to $\{c_{2n}\}$ [cf. Shih-Hsun Chang, Proc. Cambridge Philos. Soc. **48**, 87–92 (1952); these Rev. **13**, 638]. *R. P. Boas, Jr.*

Bernštein, S. N. On normally increasing weight functions and majorants of finite growth. Doklady Akad. Nauk SSSR (N.S.) **85**, 257–260 (1952). (Russian)

The author correlates a number of his recent results on weighted polynomial approximation and on inequalities for entire functions [same Doklady (N.S.) **65**, 117–120; **66**, 545–548 (1949); **77**, 549–552, 773–776 (1951); these Rev. **11**, 23; **12**, 814; **13**, 26] and one of Videnskiĭ's [see the preceding review]. In particular, even functions $\Phi(x)$ of N [normal increase: see the third reference cited above] are either in class N_1, majorizing on the real axis some even entire $F_1(z)$ with positive coefficients in its power series and not of genus zero; or in class N_0, majorized on the real axis by $F_0(z)$ of the same kind as $F_1(z)$ but of genus zero. The author then considers the class N^* of functions $\Phi(x)$ of N which belong to N_0 for $x>0$ and to N_1 for $x<0$ (in terms of an equivalent definition of these classes: $\int_1^\infty x^{-2}\log\Phi(|x|)d|x|$ converges for $x>0$ and diverges for $x<0$). He proves the following theorem. If $\int_1^\infty x^{-2}\log\Phi(x^2)dx<\infty$, or equivalently $\Phi(x)<c\prod_{n=1}^\infty(1+x/\beta_n^2)$, $c>0$, $\beta_n>0$, $\sum 1/\beta_n<\infty$, $0<x<\infty$; if $H_p(x)$ is what the author has called a function of finite semidegree [Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 111–124 (1949); these Rev. **11**, 22], i.e., $H_p(z^2)$ is an even function of exponential type p; and if $|H_p(x)|\leqq\Phi(x)$, $-\infty<x<\infty$, then

$$|H_p(x)|<2c\exp\{p|x|^{\frac{1}{2}}\}\prod_{n=1}^{\infty}(1+|x|/\beta_n^2),\quad -\infty<x<\infty,$$

irrespective of what $\Phi(x)$ is for $x<0$. *R. P. Boas, Jr.*

Boas, R. P., Jr. Growth of analytic functions along a line. Proc. Nat. Acad. Sci. U. S. A. **38**, 503–504 (1952).

Let $f(z)$ be holomorphic and of exponential type less than π in the half-plane $x\geqq 0$. Let $\lambda_n>0$, $|\lambda_n-n|<\epsilon(n)$, $n=1,2,3,\cdots$, where $\lambda_{n+1}-\lambda_n\geqq\delta>0$ while $\epsilon(x)$ is of regular growth in a certain sense; $\epsilon(x)=o(x)$. Knowing $\epsilon(x)$ and the growth of $\gamma(x)=\log|f(x)|$ on the sequence $\{\lambda_n\}$ one may ask what can be said about the growth of $\gamma(x)$ for $x>0$. For bounded $\epsilon(x)$ a very general result has been obtained by Agmon [Trans. Amer. Math. Soc. **70**, 492–508 (1951); these Rev. **12**, 815]. The present author announces results such as the following. If $\gamma(x)<K_1\epsilon(x)$ on the sequence $\{\lambda_n\}$, then $\gamma(x)<K_2\epsilon(x)$ for $x>0$. Some further results are announced for entire functions $g(z)$ of exponential type zero, among them the following best possible extension of Levinson's form of the Valiron-Pólya theorem: if $\{g(\pm\lambda_n)\}$ is bounded, and if $\int^\infty x^{-2}\epsilon(x)dx<\infty$, then $g(z)$ is a constant.

J. Korevaar (Madison, Wis.).

Boas, R. P., Jr. Integrability along a line for a class of entire functions. Trans. Amer. Math. Soc. **73**, 191–197 (1952).

Es sei $\{\lambda_n\}$ eine Folge komplexer Zahlen mit $\lambda_0=0$, $|\lambda_n-n|\leqq L$, $|\lambda_m-\lambda_n|>\delta>0$, $m\neq n$, und $f(z)$ eine ganze Funktion vom Exponentialtypus mit

$$\limsup y^{-1}\log|f(iy)f(-iy)|=2c<2\pi.$$

Dann gibt es zu gewissen wachsenden Funktionen $\phi(x)$ ($\geqq 0$) und $\beta(x)$ Konstante H und K, sodass

$$\int_{-\infty}^{+\infty}\phi(|Hf(x)|)d\beta(x)\leqq K\sum_{n=-\infty}^{+\infty}\phi(|f(\lambda_n)|)$$

(falls die Reihe auf der rechten Seite konvergiert). Dies gilt z.B. für $\beta=x$ und $\phi=x^p$, $0<p<\infty$, bzw. $\phi=e^{-1/x}$. Im Falle $\phi=x^p$, $p=2$, vgl. Duffin und Schaeffer [dieselben Trans. **72**, 341–366 (1952); diese Rev. **13**, 839] und überdies für $\lambda_n=n$ Plancherel und Pólya [Comment. Math. Helv. **9**, 224–248; **10**, 110–163 (1937)]. Für $\phi=e^{-1/x}$ ergibt sich, dass mit $\sum_{-\infty}^{+\infty}\exp(-|1/f(\lambda_n)|)$ auch das Integral

$$\int_{-\infty}^{+\infty}\exp(-|H/f(x)|)dx$$

konvergiert. Dieses Resultat war bis jetzt auch für $\lambda_n=n$ unbekannt. Der Beweis des obigen Satzes stützt sich auf die Interpolationsformel

$$f(z)=G(z)\left(\sum_{-\infty}^{+\infty}{}'\frac{f(\lambda_n)}{G'(\lambda_n)(z-\lambda_n)}\left(\frac{z}{\lambda_n}\right)^q+z^{-1}P^{-1}(z)\right),$$

$$G(z)=z\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)\left(1-\frac{z}{\lambda_{n-1}}\right),$$

die von G. Valiron unter ähnlichen Voraussetzungen bewiesen worden ist [Bull. Sci. Math. (2) **49**, 181–192 (1925)].

A. Pfluger (Zürich).

Arima, Kihachiro. On maximum modulus of integral functions. J. Math. Soc. Japan **4**, 62–66 (1952).

Let $r\theta(r,K)$ be the length of the part of $|z|=r$ where $|f(z)|>K$; $\theta(r)=\theta(r,1)$. The author proves the following theorem. If $f(z)$ is an entire function and $0<\alpha<1$, then

$$\log\log M(r)>\pi\int_{r_0}^{\alpha r}\{r\theta(r)\}^{-1}dr-c(\alpha,r_0),$$

where $0<r_0<\alpha r$ and c is independent of r. It follows that if $f(z)$ is of order ρ,

$$\rho\geqq\limsup\pi(\log r)^{-1}\int_{r_0}^{r}\{r\theta(r)\}^{-1}dr,$$

and then that for any $K>0$, $\limsup\theta(r,K)\geqq\pi/\rho$ ($\rho\geqq\frac{1}{2}$). The author also deduces a theorem of Milloux [Acta Math. **61**, 105–134 (1933)], that if $\rho<\frac{1}{2}$, then

$$\limsup\log\lambda_n/\log\rho_n\leqq(1-2\rho)^{-1},$$

where λ_n, ρ_n are the greatest and least distances from 0 to the nth island where $|f(z)|\leqq 1$. Finally the author proves that if $f(z)$ is regular in $|z|<1$ and if $\limsup\theta(r)/(1-r)<2\pi$, then either $|f(z)|<1$ or $\liminf\log\log M(r)/\{-\log(1-r)\}>0$. The proofs depend on a general result for harmonic functions which the author proves by a method of Carleman [C. R. Acad. Sci. Paris **196**, 995–997 (1933)]. *R. P. Boas, Jr.*

Iseki, Kanesiroo. On a theorem of the Phragmén-Lindelöf type. Nat. Sci. Rep. Ochanomizu Univ. **1**, 14–16 (1951).

A generalization of the Phragmén-Lindelöf theorem in the form given in Landau's Vorlesungen über Zahlentheorie, vol. 2, Th. 405 [Hirzel, Leipzig, 1927]. *J. Lehner.*

Havinson, S. Ya. On some extremal problems of the theory of analytic functions. Moskov. Gos. Univ. Učenye Zapiski **148**, Matematika **4**, 133–143 (1951). (Russian)

The results of this paper were announced in Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 158–159 (1949); these Rev. **11**, 508. The author reduces the problem of minimizing $\int_0^{2\pi}|\phi(e^{i\theta})-\omega(e^{i\theta})|^pd\theta$ (ω given, ϕ belonging to H_p with norm 1) to that of maximizing $\int_0^{2\pi}|f(e^{i\theta})\omega(e^{i\theta})|^pd\theta$ (f belonging to H_q with norm 1), and thus his problem is closely

related to the problems treated by Macintyre and Rogosinski [Acta Math. **82**, 275–325 (1950); these Rev. **12**, 89].

R. P. Boas, Jr. (Evanston, Ill.).

Hayman, W. K. Functions with values in a given domain. Proc. Amer. Math. Soc. **3**, 428–432 (1952).

Let $w=f(z)=\sum_{n=0}^{\infty}a_nz^n$ be regular in $|z|<1$ and denote by $A(R)$ the radius of the largest circle in the w-plane with center on $|w|=R$ and interior in the range of $f(z)$, $|z|<1$. For the case of univalent functions the reviewer [C. R. Acad. Sci. Paris **225**, 447–449 (1947); same Proc. **1**, 629–635 (1950); these Rev. **9**, 23; **12**, 327] obtained bounds for $|a_n|$ in terms of $A(R)$. The author, applying Schottky's theorem and a bound due to Ahlfors [Trans. Amer. Math. Soc. **43**, 359–364 (1938)] for the Bloch-Landau constant, greatly improves and generalizes the results of the reviewer. A typical result is the following: If $A(R)\leqq C$ for $0\leqq R<\infty$, then $|a_n|<eC$ $(n=1, 2, \cdots)$. If in addition $f(z)$ is univalent in $|z|<1$, then, for every $\epsilon>0$,

$$|a_n|<C(1+\epsilon)n^{-1/2}\log(6n/\epsilon)\quad(n=1, 2, \cdots).$$

A. Dvoretzky (Jerusalem).

Kobori, Akira. Une remarque sur les fonctions multivalentes. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27**, 1–5 (1952).

Let $\zeta(z)=\sum_{n=0}^{\infty}a_nz^{n-p}$, $a_0\neq 0$, be regular and p-valent in $0<|z|<1$. The author proves that $|\zeta(z)|\leqq|a_0|(4k/|z|)^p$, and $|\sum_{\nu=0}^{n}a_\nu|\leqq(1+(p/2)^{1/2})(4k)^p|a_0|$, where $k=1.03142\cdots$. The author conjectures that the first inequality is valid when $k=1$.

A. W. Goodman (Lexington, Ky.).

***Adamar, Ž. Neevklidova geometriya v teorii avtomorfnyh funkciĭ. [Hadamard, J. Non-Euclidean geometry in the theory of automorphic functions.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 134 pp. 4.95 rubles.

This is the sixth in a series of small volumes devoted to some aspect of non-euclidean geometry and its subsequent development. The present volume is a survey of classical material in the theory of automorphic functions. The six chapters are entitled "Groups of motions in the Lobatchevskian plane and their discontinuous subgroups", "Discontinuous groups in three geometries. Fuchsian groups", "Fuchsian functions", "Kleinian groups and functions", "Algebraic functions and linear algebraic differential equations", "Fuchsian groups and geodesics". The style is wholly discursive, and proofs are generally not given.

P. Davis (Washington, D. C.).

***Kober, H. Dictionary of conformal representations.** Dover Publications, Inc., New York, N. Y., 1952. xvi+208 pp. $3.95.

This dictionary, which was written for the British Admiralty, appeared in five separate volumes during 1944–1948. It has now been collected into one handy volume and it should be of considerable help to those concerned with the use of conformal maps in various branches of applied mathematics. The conformal mappings described in the book are by and large arranged according to the analytic functions giving rise to them, the author having found that this permits a more systematic classification than an arrangement according to geometric properties of domains. Part I contains a detailed discussion of numerous special cases of the linear substitution. Part II, entitled "Algebraic functions, and z^α for real α", discusses the rational function of second degree and some simple examples involving z^α; there are also some mappings by rational automorphic functions. Part III, devoted to e^z, $\log z$, and related functions, exhibits a great number of conformal maps yielded by these functions and combinations of them with those considered in Parts I and II. Part IV begins with general remarks on the Schwarz-Christoffel formula (hardly deserving the heading "On the theory of the general S.-C. transformation") and goes on to describe a large number of special cases of this formula which can be dealt with in terms of elementary functions. Section 12.9 of Part IV, entitled (according to the Table of Contents) "A converse of the method", is missing in the reviewer's copy. Part V exhibits many mappings effected by elliptic functions and also the simplest mappings associated with the elliptic modular function.

In keeping with the purely descriptive character of the book, no proofs are given. There are, however, references to the literature which may help the reader to find proofs for the statements made. The book includes an index (not quite appropriately called "Topological subject index") listing the various geometric shapes whose conformal mappings are discussed. Another valuable feature are the numerous figures and diagrams. A very useful volume.

Z. Nehari (St. Louis, Mo.).

Nagura, Shohei. Behavior of kernel functions on boundaries. Kōdai Math. Sem. Rep. **1952**, 54 (1952).

Using a theorem of Carathéodory on conformal mapping, the author shows that the Bergman and Szegö kernel functions of a domain D becomes positively infinite at any boundary point z_0 of D for which a triangle Az_0B exists in D. This enables him to extend a result of Davis and Pollak [Proc. Amer. Math. Soc. **2**, 686–690 (1951); these Rev. **13**, 337] relating the geometry of the domain with the successive derivatives of the kernel function.

P. Davis.

Fichera, Gaetano. Sulla "Kernel function". Boll. Un. Mat. Ital. (3) **7**, 4–15 (1952).

The author continues his crusade against the kernel function method [see Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 356–360, 452–457 (1951); these Rev. **13**, 931]. This time the target is Bergman's book [The kernel function and conformal mapping, Amer. Math. Soc., New York, 1950; these Rev. **12**, 402] which is taken to task for not having some of its proofs as short as the author (Fichera) can now make them. The rest of the paper is devoted to polemical remarks intended to make the reader doubt the usefulness of the kernel function concept.

Z. Nehari.

Kusunoki, Yukio. Über Streckenkomplex und Ordnung gebrochener Funktionen. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **26**, 255–269 (1951).

Simply connected covering surfaces W of the complex w-plane with a finite number of projection points of branch points are considered. Let $z(w)$ be the analytic function mapping W onto the disc $|z|<1$ or $|z|<\infty$. Under certain restrictions imposed upon the line complex of W, the author gives estimates for the order of $w(z)$, in terms of the line complex.

L. Sario (Cambridge, Mass.).

***Volkovyskiĭ, L. I. Investigation of the type problem for a simply connected Riemann surface.** Trudy Mat. Inst. Steklov. **34**, 171 pp. (1950). (Russian)

This extensive paper represents the complete doctoral dissertation of the author, the original parts of which had

been previously published in several shorter papers [Mat. Sbornik N.S. **18**(60), 185–212 (1946); **23**(65), 229–258, 361–382 (1948); Uspehi Matem. Nauk **3**, no. 3(25), 215–216 (1948); these Rev. **8**, 326; **10**, 364, 365]. Some of the material is expository in character and discusses the work of Grötzsch, Ahlfors, Kobayashi, Blanc, and others. There are more detailed examples and applications than in the earlier papers. *W. Seidel* (Princeton, N. J.).

Yosida, Tokunosuke. On a sufficient condition for a given Riemann surface to be of hyperbolic type. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 89–92 (1941).

The author considers simply-connected covering surfaces of the complex plane, subject to the following specific restrictions: (1) all branch points are logarithmic; (2) there are only three projection points of branch points; (3) the topological tree of the surface has no logarithmic ends. The knots from which three disjoint lines originate are arranged in generations. Let $\varphi(n)$ and $\varphi'(n)$ be the largest and smallest number of simplexes joining knots of $(n-1)$th and nth generations. Denote by $\{t_n\}$ a sequence of positive numbers with bounded values of $\sum_{i=1}^{n-1}t_i/t_n$ and t_{n+1}/t_n. If the sums $\sum\varphi(n)/2^n t_n$, $\sum t_n/2^n\varphi'(n)$ are both convergent, then the surface is of hyperbolic type. In the proof, use is made of Kakutani's representation method [Jap. J. Math. **13**, 375–392, 393–404 (1937)]. *L. Sario* (Cambridge, Mass.).

Kobayashi, Zen-ichi. On Kakutani's theory of the type of Riemann surfaces. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. **4**, 9–44 (1940).

This is an investigation of the type problem in its classical sense: conformal equivalence of a given simply-connected Riemann surface with the punctured plane or with the unit circle. The paper deals with the special class of covering surfaces of the complex plane with a finite number of projection points of branch points. The author makes use of Kakutani's representation of these surfaces [Jap. J. Math. **13**, 375–392, 393–404 (1937)] and of Teichmüller's theory of quasi-conformal mapping [Deutsche Math. **3**, 621–678 (1938)].

The principal results are as follows. (1) The type is invariant under certain topological transformations of Speiser's line complex of the surface. (2) Let $\mu(n)$ be the number of knots in the nth generation. If the surface is "almost homogeneous", in a specified sense, and has no algebraic branch points, then a sufficient condition for hyperbolic type is the convergence of $\Sigma n/\mu(n)$. (3) Consider a transformation $F(T_0)=T$ of the line complex. If the surface represented by T_0 is almost homogeneous, then the convergence of $\Sigma n\psi(n)/\mu_0(n)$ is sufficient for the surface associated with T to be of hyperbolic type. Here $\psi(n)$ characterizes the degree of deformation of the nth subcomplex of T_0. (4) If the surface is almost homogeneous and fulfills Kakutani's condition [loc. cit.], then the type is parabolic if and only if $\sum 1/n\mu(n)$ diverges. (5) Let $\sigma(n)$ be the number of free knots in the nth generation. There are special classes of almost homogeneous surfaces for which the divergence of Wittich's series $\sum 1/\sigma(n)$ [Math. Z. **45**, 642–668 (1939); these Rev. **1**, 211] is a necessary and sufficient condition for the parabolic type. *L. Sario* (Cambridge, Mass.).

Tsuji, Masatsugu. On the uniformization of an algebraic function of genus $p\geqq 2$. Tôhoku Math. J. (2) **3**, 277–281 (1951).

Consider a closed covering surface F of the complex x-plane, of genus $p\geqq 2$. The author establishes the following theorem. There does not exist any meromorphic function $x=x(t)$ in a neighborhood of a closed point set E of logarithmic capacity zero such that each point of E is an essential singularity of $x(t)$ and such that the Riemann surface created by $x(t)$ is a relatively non-ramified covering surface of F. The proof is based on Ahlfors' theory of covering surfaces [Acta Math. **65**, 157–194 (1935)]. Then the author, correcting his earlier erroneous reasoning [Jap. J. Math. **18**, 759–775, 977–984 (1943); **19**, 155–188 (1944); these Rev. **7**, 516], shows that the Schottky point set has positive capacity [Myrberg, Ann. Acad. Sci. Fennicae Ser. A. I. Math.-Phys. no. **10** (1941); these Rev. **7**, 516]. *L. Sario* (Cambridge, Mass.).

Motzkin, T. S., and Schoenberg, I. J. On lineal entire functions of n complex variables. Proc. Amer. Math. Soc. **3**, 517–526 (1952).

A polynomial $P(z)=P(z_1,\cdots,z_n)$ with complex coefficients is called lineal if it is a product of linear functions. An entire function $f(z)$ is called lineal if it is the limit of a sequence of lineal polynomials uniformly convergent on every compact set. The author proves that $f(z)$ is lineal if and only if it vanishes on planes only. If $f(z)$ is lineal, it can be written as a Weierstrass product

$$f(z)=\exp G(z)\prod_1^n(z,c_j)\prod_1^\infty P_k(z,d_k)$$

where $G(z)$ is an entire function, c_j and d_k complex vectors, and

$$P_k(u)=(1-u)\exp\left(u+\tfrac{1}{2}u^2+\cdots+\frac{1}{k-1}u^{k-1}\right).$$

The function $f(z)$ is called really lineal if the approximating polynomials split into products of linear functions with real coefficients. This happens if and only if

$$f(z)=\exp\left(-\sum_1^m c_{rs}z_rz_s+(z,d)\right)\prod_1^n(z,c_j)\prod_1^\infty(1+(z,d_k))e^{-(z,d_k)},$$

where c_j, d and d_k are real vectors, $\sum\|d_k\|^2$ convergent and $\sum_1^m c_{rs}z_rz_s\geqq 0$ for every real z. Finally $f(z)$ is called positively lineal if the approximating polynomials split into linear factors with positive coefficients. In this case the vectors c_j, d, and d_k have positive coordinates, $\sum\|d_k\|$ converges and the quadratic form vanishes. Detailed proofs of these statements are given. *H. Tornehave* (Lyngby).

San Juan, Ricardo. Une propriété générale des classes quasi analytiques et des développements asymptotiques dans des demi-plans. C. R. Acad. Sci. Paris **235**, 282–284 (1952).

The author shows that every quasi-analytic class $C\{M_n\}$, on a finite or infinite interval, is contained in a quasi-analytic class $C\{M_n^*\}$ with M_n^* non-decreasing; for the interval $(-\infty,\infty)$ this is due to Gorny [Acta Math. **71**, 317–358 (1939), p. 329; these Rev. **1**, 137, 400]. Similarly, if $\Phi(z)$ is regular for $\Re(z)>0$ and has an asymptotic expansion with bounds m_n, i.e.,

$$\Phi(z)=\sum_{\nu=0}^{n-2}a_\nu z^{\nu+1}+z^n\Phi_n(z),$$

with $|\Phi_n(z)|<m_n$, then in every half-plane $\Re(z)\geqq a>0$ it has the same expansion with bounds $m_n^*<c^n m_{n+2}$ and m_n^* nondecreasing. *R. P. Boas, Jr.* (Evanston, Ill.).

Lalaguë, Pierre. Sur certaines classes de fonctions indéfiniment dérivables. C. R. Acad. Sci. Paris **235**, 114–116 (1952).

The author considers classes of infinitely differential functions where certain norms are introduced. The problem of inclusion of two classes in the sense of Mandelbrojt is solved for these classes. As is to be expected the Laguerre polynomials play a rather important role in the discussion.

S. Agmon (New York, N. Y.).

Theory of Series

Krishnamoorthy, A. S. On the ratios of one term to the partial sums in a divergent series of positive terms. Math. Student **19** (1951), 102–104 (1952).

If $0<\alpha_n<1$, $\sum\alpha_n=\infty$, and $d_1>0$, then there is a unique divergent series $\sum d_n$ such that $d_n/D_n=\alpha_n$ when $D_n=\sum_{j=1}^n d_j$ and $n>1$. When k is a positive integer and one of $\lim_{n\to\infty}\alpha_n$ and $\lim_{n\to\infty} d_n/D_{n+k}$ exists, then the other exists and the two are related by simple formulas.

R. P. Agnew.

Obrechkoff, Nikola. Sur la convergence des séries. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 327–342 (1950). (Bulgarian. French summary)

Several theorems on convergence of series are given. While they are closely related to familiar facts and are easily proved, the author says they are new. Some examples follow. Let $u_1+u_2+u_3+\cdots$ be a series of constants for which $\lim u_n=0$. Let $\lambda_1, \lambda_2, \cdots, \lambda_p$ be given constants, let $\lambda=\lambda_1+\lambda_2+\cdots+\lambda_p$, and let

$$w_n=\lambda_1u_{n+1}+\lambda_2u_{n+2}+\cdots+\lambda_pu_{n+p}.$$

If the numbers u_n are real, if $\lambda>0$, and if $w_n\geqq 0$ when $n\geqq N$, then $\sum u_n$ converges or diverges to $+\infty$. If the hypotheses are the same except that $\lambda<0$, then the series converges or diverges to $-\infty$. Existence of one set of λ's satisfying the first set of conditions, and another set of λ's satisfying the second set, implies convergence of $\sum u_n$. If the numbers u_n and λ_n are complex, if $\lambda\neq 0$, and if all of the points w_n for which $n>N$ lie in a sector of the complex plane with vertex at the origin and angle less than π, then either $\sum u_n$ converges or $\sum u_n$ has partial sums s_n for which $|s_n|\to\infty$.

Let $\sum u_n$ and $\sum v_n$ be series of real (but not necessarily nonnegative) terms and let $\sum u_n$ converge. If $v_n\leqq u_n$, then $\sum v_n$ converges or diverges to $-\infty$; if $v_n\geqq u_n$, then $\sum v_n$ converges or diverges to $+\infty$. If $v_n\to 0$ and $\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q$ are real constants such that

$$\alpha_1v_{n+1}+\alpha_2v_{n+2}+\cdots+\alpha_pv_{n+p}\leqq\beta_1u_{n+1}+\beta_2u_{n+2}+\cdots+\beta_qu_{n+q}$$

when $n>N$ and if $\alpha_1+\cdots+\alpha_p>0$, then $\sum v_n$ converges or diverges to $-\infty$.

R. P. Agnew (Ithaca, N. Y.).

Gandini, Carla. Un teorema di A. E. Ingham sui "Grandi indici". Boll. Un. Mat. Ital. (3) **7**, 143–148 (1952).

A method of G. Ricci [Ann. Mat. Pura Appl. (4) **13**, 287–308 (1935)] is applied to prove the following big-gap theorem of Ingham [Quart. J. Math., Oxford Ser. **8**, 1–7 (1937)]. If $0<\lambda_1<\lambda_2<\cdots$, if $\lambda_{n+1}/\lambda_n\to\infty$, and if $\sum a_n$ is a real series for which $\sum a_n \exp(-\lambda_n s)$ converges for $s>0$ to a function $f(s)$ having finite inferior and superior limits as $s\to 0$, then

$$\overline{\underline{\lim}}_{n\to\infty}(a_1+a_2+\cdots+a_n)=\overline{\underline{\lim}}_{s\to 0} f(s).$$

R. P. Agnew (Ithaca, N.Y.).

Karadžić, Lazar. Sur un théorème inverse-0. Bull. Soc. Math. Phys. Serbie **3**, nos. 3–4, 25–36 (1951). (Serbo-Croatian. French summary)

A series $\sum a_n$ converges to s if

$$\lim_{y\to\infty}\left[a_0+\sum_{n=1}^{\infty}(\Delta^{n-1}a_1)y^n/n!\right]=s$$

and two specified Tauberian conditions, restricting respectively the magnitudes of $\Delta^{n-1}a_1$ and a_n, are satisfied.

R. P. Agnew (Ithaca, N. Y.).

Zeller, K. Verallgemeinerte Matrixtransformationen. Math. Z. **56**, 18–20 (1952).

The author proves very simply by means of functional analysis the generalization of the theorem of Toeplitz on methods of summation (given by A. Robinson [Proc. London Math. Soc. (2) **52**, 132–160 (1950); these Rev. **12**, 253] and Melvin-Melvin [ibid. (2) **53**, 83–108 (1951); these Rev. **13**, 45]), in which the complex matrix (a_{mn}) is replaced by a matrix of continuous linear operators (A_{mn}). He asks for a simple proof of the variation of the above theorem (due to Robinson) in which the A_{mn} are not assumed to be continuous. The answer to this is given in a forthcoming independent paper by Macphail and the reviewer.

G. G. Lorentz (Toronto, Ont.).

Zeller, Karl. Faktorfolgen bei Limitierungsverfahren. Math. Z. **56**, 134–151 (1952).

Necessary and sufficient conditions for a matrix $A=(a_{nk})$ are given in order that there is a sequence of integers $q_1<q_2<\cdots$ such that each sequence $s_n=u_0+u_1+\cdots+u_n$ with the properties (a) $U_i=\sum_{q_i<k\leqq q_{i+1}}|u_k|=o(1)$ or (b) $U_i=O(1)$ or (c) $M_i=\max_{q_i<k,\, l\leqq q_{i+1}}|s_k-s_l|=o(1)$ or (d) $M_i=O(1)$ be A-summable. For (b) these are for instance $\lim_r[\sup_n|\sum_{k=r}^{\infty}a_{nk}|]=0$ and the existence of $\lim_n a_{nk}$, $k=0, 1, \cdots$; for (c), $\sum_k|a_{nk}|=O(1)$ and

$$\chi=\lim_n\sum_k a_{nk}-\sum_k\lim_n a_{nk}=0.$$

An A-summable sequence x_k is said to belong to $\mathfrak{A}_1$ or $\mathfrak{A}_2$ or $\mathfrak{A}_3$ if the matrix $G=(a_{nk}x_k)$ sums each absolutely convergent sequence, or satisfies conditions for (a) or for (b), respectively. Properties of sequences from $\mathfrak{A}_i$ are given, describing their maximal order of increase, approximation by finite sequences, and invariance with respect to some multiplicator sequences. The author also proves in a generalized form some theorems of Mazur and Orlicz [C. R. Acad. Sci. Paris **196**, 32–34 (1933)] and Agnew [Ann. of Math. (2) **46**, 93–101 (1945); these Rev. **6**, 150].

G. G. Lorentz.

Jurkat, W., and Peyerimhoff, A. Mittelwertsätze und Vergleichssätze für Matrixtransformationen. Math. Z. **56**, 152–178 (1952).

This paper continues earlier work of the authors [Math. Z. **55**, 92–108 (1952); these Rev. **13**, 934]. Triangular methods of summation $A=(a_{mn})$ are considered which satisfy one of the two mean value theorems:

$$(\mathfrak{M})\qquad \left|\sum_{\nu=0}^{n}a_{m\nu}s_\nu\right|\leqq K\left|\sum_{\nu=0}^{n'}a_{n'\nu}s_\nu\right|,$$

$$(\mathfrak{M}^*)\qquad \left|\sum_{\nu=0}^{n}a_{m\nu}s_\nu\right|\leqq K^*\sum_{\nu=0}^{n}|a_{m\nu}|\left|\sum_{\nu=0}^{n'}a_{n'\nu}s_\nu\right|\Big/\sum_{\nu=0}^{n'}|a_{n'\nu}|,$$

The reviewer wishes to retract his statement in the last four lines of the review that the hypothesis that the methods of summation are perfect could have been avoided in the paper.

where K, K^* are constants, $0\leqq n'\leqq n\leqq m$, $\{s_\nu\}$ an arbitrary sequence, and n' depends on n, m and $\{s_\nu\}$. $\mathfrak{M}$ and $\mathfrak{M}^*$ hold if $0\leqq a_{m\nu}/a_{n\nu}\leqq K$ and $a_{m\nu}/a_{n\nu}$ is a decreasing function of ν $(0\leqq\nu\leqq n\leqq m)$. The following general inclusion theorems are proved. Let A, B be triangular methods with $b_{n\nu}\neq 0$ for $\nu\leqq n$, $a_{n\nu}\to 0$, $b_{n\nu}\to 0$ for $n\to\infty$, and let

$$(1)\qquad \sum_{\nu=0}^{n}|(a_{n\nu}/b_{n\nu})-(a_{n,\nu+1}/b_{n,\nu+1})|=O(1).$$

If the boundedness of the B-transform of a sequence implies the boundedness of its A-transform, then $A\subseteq B$. The same conclusion holds if $\mathfrak{M}$ is replaced by $\mathfrak{M}^*$ and (1) by $\sum_{\nu=\mu}^{n}|(a_{n\nu}/b_{n\nu})-(a_{n,\nu+1}/b_{n,\nu+1})|=O(a_{n\mu}/b_{n\mu}+1)$, $0\leqq\mu\leqq n$. This is applied to deduce several inclusion and equivalence theorems, mostly known, of the methods of arithmetic means, Nörlund methods and Riesz methods of order $0<\kappa\leqq 1$. [The reviewer remarks that the use of properties of perfect methods could have been avoided, since the restriction that B be perfect in Satz 3 can be dropped if A is assumed triangular.] *G. G. Lorentz* (Toronto, Ont.).

Love, E. R. Mercer's summability theorem. J. London Math. Soc. **27**, 413–428 (1952).

The author shows that if the method $C=(c_{nk})$ preserves convergence, $s_n-q\sum_{k=1}^{\infty}c_{nk}s_k$ converges for $n\to\infty$, s_n is bounded and $|q|<1/(N-\sum_{k=1}^{\infty}c_k)$ with $N=\limsup\sum_k|c_{nk}|$, $c_k=\lim_n|c_{nk}|$, then s_n converges. Other Mercerian theorems of the same type given by the author (in particular, Corollary 1, p. 416 and Theorem 4) and the case $c_k=0$ of the above theorem were also proved in a paper by R. Rado [Quart. J. Math., Oxford Ser. 9, 274–282 (1938)] which is not mentioned. Finally, a Mercerian theorem for absolute summability is given which generalizes a result by Bosanquet [J. London Math. Soc. 13, 177–180 (1938)]. This paper continues the work of Leslie and the author [Proc. Amer. Math. Soc. 3, 448–457 (1952); these Rev. 13, 836].

G. G. Lorentz (Toronto, Ont.).

Pennington, W. B. Some inequalities related to Abel's method of summation. Proc. Amer. Math. Soc. **3**, 557–565 (1952).

The following and related results are given. Let $\sum a_n$ be a real series for which $\sum a_nx^n$ converges when $0<x<1$. Let $r>0$, let $p\geqq 1$, and let

$$x=\exp(-ru^{-1}(\log u)^{-p})$$

when $u>1$. Then

$$\liminf_{n\to\infty}\left\{\sum_{n=0}^{\infty}a_nx^n-x^u\sum_{n\leqq u}a_n\right\}\geqq p\liminf_{n\to\infty}a_nn\log\log n.$$

Moreover the factor p is the least factor for which the hypotheses imply the conclusion. *R. P. Agnew.*

Chadaya, T. G. Summability of double series by Nørlund's method. Soobščeniya Akad. Nauk Gruzin. SSR. **11**, 143–146 (1950). (Russian)

Let N be a regular Nörlund double-sequence transformation with positive coefficients. It is shown that if S_{mn} is convergent to S and has rows and columns satisfying stated conditions, then the N transform of S_{mn} converges restrictedly (in the sense of C. N. Moore) to S.

R. P. Agnew (Ithaca, N. Y.).

Čelidze, V. G. Borel summation of double series. Soobščeniya Akad. Nauk Gruzin. SSR. **8**, 501–508 (1947). (Russian)

Čelidze, V. G. Linear transformations of double numerical sequences. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 333–339 (1948). (Russian)

Čelidze, V. G. On generalized Abel summability of double series. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 457–462 (1948). (Russian)

Čelidze, V. G. The summability of double series. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **16**, 1–37 (1948). (Georgian. Russian summary)

These four papers treat special problems, in the theory of transformations of double sequences, that arise because of the following fact. A double sequence s_{jk} may be convergent to s in the sense of Pringsheim (that is, $\lim_{j,k\to\infty}s_{j,k}=s$) and nevertheless have a nonempty finite set of unbounded rows and columns. Let A be a sequence-to-sequence transformation of the form $\sigma_{mn}=\sum_{j,k=0}^{\infty}a_{m,n,j,k}s_{jk}$ or a sequence-to-function transformation $\sigma(t,u)=\sum_{j,k=0}^{\infty}a_{j,k}(t,u)s_{jk}$. In each paper A is one of the transformations of Cesàro, Abel, and Borel, or is a more general transformation satisfying specified conditions. The results are of the following type. Let s_{jk} converge to s, and let each row and column of s_{jk} satisfy a stated condition (depending upon the transformation A being considered) which restricts the absolute values of the elements but does not require that they be uniformly bounded. It is then shown that the transform must converge restrictedly (in the sense of C. N. Moore) to s. The conclusion means, in the case of a sequence transform $\sigma_{m,n}$, that if $\lambda>1$ and $\epsilon>0$, then there is an index N such that $|\sigma_{m,n}-s|<\epsilon$ whenever $m,n>N$ and $\lambda^{-1}<m/n<\lambda$. In the case of the function transform, the meaning is analogous.

R. P. Agnew (Ithaca, N. Y.).

Marmarašvili, G. A. Cesàro summability of functions of two variables. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 273–276 (1948). (Russian)

Čelidze, V. G. On double transformations of functions of two variables. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 521–525 (1948). (Russian)

Magnaradze, Leo. Direct and inverse limit theorems for double integral transformations. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 527–532 (1948). (Russian)

These three papers treat special and more general function-to-function kernel transformations of the type

$$F(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(x,y,u,v)f(u,v)dudv.$$

The problems are analogous to the problems of the four preceding papers. If $\lim_{u,v\to\infty}f(u,v)=s$ and if $f(u,v)$ satisfies appropriate additional conditions depending upon K, then $F(x,y)$ converges restrictedly to s as $x,y\to\infty$.

R. P. Agnew (Ithaca, N. Y.).

Mazur, S. On the generalized limit of bounded sequences. Colloquium Math. **2** (1951), 173–175 (1952).

The author gives an alternative proof of the existence of the generalized limit of a bounded sequence due to Banach [Théorie des opérations linéaires, Warsaw, 1932, p. 34] and himself [First Polish Mathematical Congress, Lwów, 1927, Supplément Ann. Soc. Polon. Math. 1929, p. 103] based on properties of Tychonoff's cube instead of the extension theorem for linear functionals. *J. D. Hill.*

Lorentz, G. G. Riesz methods of summation and orthogonal series. Trans. Roy. Soc. Canada. Sect. III. (3) 45, 19–32 (1951).

This paper contains a discussion of Riesz summability $R(\lambda_n, 1)$ of orthogonal series $\sum a_n\phi_n(x)$, $\sum a_n^2<+\infty$, where the $\phi_n(x)$ are orthonormal on an interval (a, b). Since the results are too numerous and detailed to quote at length we mention merely a few that are typical. In Section 1, where known results concerning C_k-summability are generalized, it is shown, for example, that the condition $\sum a_n^2(\log\log\lambda_n)^2<+\infty$ implies the $R(\lambda_n, 1)$ summability a.e. of $\sum a_n\phi_n(x)$. This result includes the corresponding results of Menchoff-Rademacher ($\lambda_n=e^n$) and Menchoff-Kaczmarz ($\lambda_n=n$). Among the applications of Section 1 given in Section 2, we find the following results, each a consequence of the result just stated. (i) For any sequence $\{a_n\}$ with $\sum a_n^2<+\infty$ there is a method $R(\lambda_n, 1)$ which sums every orthogonal series $\sum a_n\phi_n(x)$. (ii) For any orthogonal system $\{\phi_n(x)\}$ there is a method $R(\lambda_n, 1)$ effective for all series $\sum a_n\phi_n(x)$. In Section 3 the author discusses the inclusion of $R(\lambda_n, 1)$ in a general method A and applies the results in Section 4 to obtain generalizations of some of his earlier theorems. He shows, for instance, that if $A=(a_{mn})$ is strongly regular, i.e., if $\sum_n|a_{mn}-a_{m,n+1}|\to 0$, there is a function $\omega(n)=o(\log^2 n)$ such that $\sum a_n^2\omega(n)<+\infty$ implies the A-summability a.e. of $\sum a_n\phi_n(x)$. *J. D. Hill.*

Sargent, W. L. C. On the summability of infinite integrals. J. London Math. Soc. 27, 401–413 (1952).

Several lemmas on linear transformations culminate in the following theorem on Cesàro generalizations of $\lim_{A\to\infty}\int_a^A x(t)dt$. If $a>0$ and $\lambda\geqq 0$, then a function $k(t)$ is such that $\int_a^\infty x(t)k(t)dt$ is evaluable (C, λ), whenever $\int_a^\infty x(t)dt$ is evaluable (C, λ), if and only if there exists a number $b\geqq a$ such that $k(t)$ is measurable and essentially bounded over $a\leqq t\leqq b$ and $k(t)$ is equivalent over $t\geqq b$ to a function $\theta(t)$ of the form

$$\theta(t)=L+[\Gamma(\lambda+1)]^{-1}\int_t^\infty (u-t)^\lambda d\alpha(u)$$

where L is a constant and $\int_b^\infty t^\lambda|d\alpha(t)|<\infty$.
R. P. Agnew (Ithaca, N. Y.).

Agnew, Ralph Palmer. Arithmetic means of some Tauberian series and determination of a lower bound for a fundamental Tauberian constant. Proc. London Math. Soc. (3) 2, 369–384 (1952).

Let $s_n=u_0+\cdots+u_n$, $M_n=(s_0+\cdots+s_n)/(n+1)$ and $\limsup|nu_n|<+\infty$. This paper discusses Tauberian constants B for which (1) $\limsup|M_n-s_{p_n}|\leqq B\limsup|nu_n|$ holds for some sequence $p_n\to\infty$. If p_n is not allowed to depend on u_n, the least B for all possible p_n is $\log 2$ (as obtained by the author in an unpublished paper). The corresponding problem for Abel's summability was also solved by the author [Trans. Amer. Math. Soc. 72, 501–518 (1952); these Rev. 13, 934]. Here the more delicate problem is attacked to determine the least possible B when the p_n in (1) are allowed to depend on the u_n. In this case B in (1) is connected with the unique root $B_0=0.474541$ of the equation $\exp(-\frac{1}{2}\pi x)=x$: In the above situation, $B\geqq B_0$ if for each bounded sequence s_n there is a sequence $p_n\to\infty$ with (2) $\limsup|M_n-s_{p_n}|<B\limsup|nu_n|$; and $B>B_0$ if this is true without the restriction that s_n is bounded. It is conjectured that (2) is true for $B\geqq B_0$ and $B>B_0$, respectively. Examples required for the proofs depend on a study of curves C_h defined as follows. Let $\lim|nu_n|=h>0$ and let the s_n lie on a closed curve C; then C_h is the set of all limit points of the M_n. *G. G. Lorentz* (Toronto, Ont.).

Rajagopal, C. T. Two one-sided Tauberian theorems. Arch. Math. 3, 108–113 (1952).

Unilateral Tauberian theorems proved by Karamata [Comment. Math. Helv. 25, 64–69 (1951); these Rev. 12, 694] and others are generalized from the Abel power-series transformation to a class of transformations of the form $\Phi(t)=\sum_{n=1}^\infty a_n\varphi(\lambda_n t)$. *R. P. Agnew* (Ithaca, N. Y.).

Rajagopal, C. T. On a one-sided Tauberian theorem. J. Indian Math. Soc. (N.S.) 16, 47–54 (1952).

This paper and another [see the preceding review], written later by the same author, have the same purpose.
R. P. Agnew (Ithaca, N. Y.).

Knopp, Konrad. Zwei Abelsche Sätze. Acad. Serbe Sci. Publ. Inst. Math. 4, 89–94 (1952).

The author proves Abelian theorems for Laplace and Abel transforms which are akin to well-known Tauberian theorems of Karamata. Let $L(t)$ be positive and continuous over $t>0$, and be slowly varying in the sense that if $u>0$, then $L(ut)/L(t)\to 1$ as $t\to\infty$. The first theorem says that if $s(t)$ is measurable and bounded over each finite interval $0\leqq t\leqq t_0$ and if $s(t)/t^\alpha L(t)\to s$ as $t\to\infty$, then

$$\frac{1}{y}\int_0^\infty e^{-t/y}s(t)dt/y^\alpha L(y)\to s\Gamma(\alpha+1)$$

as $y\to\infty$. The second theorem, in the conclusion of which k is misprinted for n and the last n should be $1/(1-x)$, should be the following: If $s_n/n^\alpha L(n)\to s$, then

$$(1-x)\sum_{n=0}^\infty s_n x^n/(1-x)^{-\alpha}L(1/(1-x))\to s\Gamma(\alpha+1)$$

as $x\to 1-$. *R. P. Agnew* (Ithaca, N. Y.).

Scherrer, W. Elementare Bestimmung der Summe der reziproken Quadratzahlen. Elemente der Math. 7, 103–106 (1952).

Banditch, Ivan M. Geometrische Interpretation eines Satzes von L. Euler. Bull. Soc. Math. Phys. Serbie 3, nos. 3–4, 45–48 (1951). (Serbo-Croatian. German summary)

The formula in question is $\cos 2^{-1}\omega\cdot\cos 2^{-2}\omega\cdots=\omega^{-1}\sin\omega$.

Koschmieder, Lothar. Bounds for partial sums of some series important in the theory of elliptic functions. Univ. Nac. Tucumán. Revista A. 8, 89–105 (1951). (Spanish)

Let the function $F(x)$ have a continuous derivative $F'(x)$ in $(0, 1)$; let $F(x)$ and $F'(x)$ satisfy one of the four sets of conditions: (a) $F(0)=F(1)=0$; (b) $F(0)=F'(1)=0$; (c) $F'(0)=F(1)=0$; (d) $F'(0)=F'(1)=0$, $\int_0^1 F(\alpha)d\alpha=0$; define for each of the four cases, respectively, (a) $\varphi_m(x)=2\sin m\pi x$; (b) $\varphi_m(x)=2\sin(m-\frac{1}{2})\pi x$; (c) $\varphi_m(x)=2\cos(m-\frac{1}{2})\pi x$; (d) $\varphi_m(x)=2\cos m\pi x$ and set $\beta_m=\int_0^1 F(\alpha)\varphi_m(\alpha)d\alpha$. Then [see Kneser, Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik, Vieweg, Braunschweig, 1911] the expansions (*) $F(x)=\sum_{m=1}^\infty\beta_m\varphi_m(x)$ hold. Assuming, furthermore, that $F''(x)<0$ in $(0, 1)$ in the cases (a), (b), (c) and $-c<F''(x)<C$ in the case (d), the following bounds for the partial sums $S_n(x)$ of the series (*) are quoted from an earlier paper of the author [Monatsh. Math. Phys. 39,

321–344 (1932)]:

(a) $0<S_n(x), \quad S_{2n}(x)<\frac{1}{2}\{F'(0)-F'(1)\}$;

(b) $0<S_n(x)<F'(0)$;

(c) $0<S_n(x)<-F'(1)$;

(d) $-c/6<S_{2n}(x)<c/3, \quad -C/3<S_{2n}(x)<C/6$.

Using these results, bounds are found for the partial sums of series (*), corresponding to some elementary functions, to the elliptic functions sn u, cn u, dn u, am $u-\pi u/2K$, $Z(u)$ [see Greenhill, The applications of elliptic functions, Macmillan, London, 1892] and to the theta functions $\vartheta_1(x)$, $\vartheta_2(x)$, $\vartheta_3(x)$, and $\vartheta_0(x)$. *E. Grosswald.*

Tanaka, Chuji. Note on Dirichlet series. III. On the singularities of Dirichlet series. III. Tôhoku Math. J. (2) 4, 49–53 (1952).

In the present note the author generalizes the fundamental theorem of his first note [same J. (2) 3, 285–291 (1951); these Rev. 13, 832] by allowing the existence of an exceptional set of exponents $\{\lambda_{n_k}\}$ which may be suppressed in formulating the conditions. He calls $\{\lambda_{n_k}\}$ a normal subsequence of density zero if (1) $\lim k/\lambda_{n_k}=0$, (2) $\liminf (\lambda_{n_k}-\lambda_{n_{k-1}})>0$, and (3) $\liminf |\lambda_{n_k}-\lambda_n|>0$ if k, $n\to\infty$ but $n\neq n_k$. The terms corresponding to such a subsequence may be disregarded in verifying whether or not the conditions of the theorems proved by the author in his previous notes are satisfied. *E. Hille* (Nancy).

Tanaka, Chuji. Note on Dirichlet series. IX. Remarks on J. J. Gergen-S. Mandelbrojt's theorems. Proc. Japan Acad. 28, 73–76 (1952).

The theorems referred to in the title appeared in Amer. J. Math. 53, 1–14 (1931). The author proves that if the Dirichlet series $F(s)=\sum_1^\infty a_n \exp(-\lambda_n s)$ converges in the whole plane, and if $F(s)$ is bounded for $\sigma\geqq\sigma_0-\delta$ (in particular, if the series is uniformly convergent), then $F(s)$ takes on every value with at most two exceptions, ∞ included, in every double sector $|\arg(s-s_0)\pm\pi/2|<\epsilon$, $s_0=\sigma_0+it_0$. *E. Hille* (Nancy).

Fourier Series and Generalizations, Integral Transforms

Šapiro-Pyateckiĭ, I. I. On uniqueness of expansion of a function in trigonometric series. Doklady Akad. Nauk SSSR (N.S.) 85, 497–500 (1952). (Russian)

Let P be a perfect nondense set situated on an interval ρ (by ρ we shall also denote the length of the interval) and let $d_1, d_2, \cdots, d_n, \cdots$ be the intervals contiguous to P. Let us call d_1/ρ the "index" of d_1. Suppose that the indexes of the intervals $d_1, d_2, \cdots, d_n$ have already been determined. After the removal of these intervals from ρ, there remain $n+1$ segments; let δ denote that segment which contains d_{n+1}. Then by the index of the interval d_{n+1} we mean the ratio d_{n+1}/δ. N. Bary proved [Fund. Math. 9, 62–115 (1927), pp. 84–100] that, if $\rho=(0, 2\pi)$, the set P is a set of multiplicity for trigonometric series (i.e., there is a series which converges to 0 outside P, mod 2π, but is not identically zero) provided the sequence of intervals $d_1, d_2, \cdots$ can be rearranged in such a way that the following conditions are satisfied: 1) the index of d_n tends to 0 as $n\to\infty$; 2) the ratio of the larger to the smaller of the two intervals which are obtained by removing d_{n+1} from δ (see above) stays below a finite number independent of n. She also conjectured that condition 2) is superfluous. S. Verblunsky [Acta Math. 65, 283–305 (1935)] gave a proof of this conjecture. In the present note the author shows that Verblunsky's proof is not correct and that the result itself is false: There exist perfect sets of uniqueness satisfying condition 1). [The incorrectness of Verblunsky's proof (but not of the result) was also noticed by Civin and Chrestenson [Bull. Amer. Math. Soc. 58, 567 (1952)].] *A. Zygmund* (Chicago, Ill.).

Men'šov, D. E. On Fourier series of continuous functions. Moskov. Gos. Univ. Učenye Zapiski 148, Matematika 4, 108–132 (1951). (Russian)

In a previous paper [Mat. Sbornik N.S. 11(53), 67–96 (1942); these Rev. 7, 59] the author showed that any continuous function $f(x)$ of period 2π can be changed in a set e of measure less than a prescribed $\epsilon>0$ in such a way that the Fourier series of the resulting function converges uniformly. The set e here depends both on $f(x)$ and ϵ. The author now shows that, if ϵ is fixed and the modulus of continuity of f does not exceed a fixed function $\rho(\delta)$ tending to 0 with δ, then the set e may be chosen independently of f. Thus $e=e(\epsilon, \rho(\delta))$. He also shows that the dependence on $\rho(\delta)$ is essential, that is, e cannot depend on ϵ only. This follows from the following result. For every perfect set $P\subset(0, 2\pi)$ of positive measure and for every point x_0 of density 1 for P, one can find a function $f(x)$ continuous on $(0, 2\pi)$ and such that no matter what function $\varphi(x)$ we consider which is bounded on $(0, 2\pi)$ and coincides with $f(x)$ on P, the Fourier series of $\varphi(x)$ diverges at x_0. *A. Zygmund* (Chicago, Ill.).

Men'šov, D. E. On Fourier series of summable functions. Trudy Moskov. Mat. Obšč. 1, 5–38 (1952). (Russian)

(1) Let $f(x)$ be any function integrable over the interval $0\leqq x\leqq 2\pi$, and let P be any perfect non-dense set on that interval. The function f can then be changed, in the complement of P, in such a way that the resulting function $g(x)$ is integrable and its Fourier series converges almost everywhere. In a previous paper [Mat. Sbornik N.S. 11(53), 67–96 (1942); these Rev. 7, 59] the author showed that (2) f can be changed in an unspecified set e of arbitrarily small measure in such a way that the Fourier series of the new function converges uniformly. He had also shown [Doklady Akad. Nauk SSSR 67, 787–789 (1949); these Rev. 11, 26] that, in (1), convergence almost everywhere cannot be replaced by convergence everywhere (and hence not by uniform convergence). *A. Zygmund* (Chicago, Ill.).

Lukacs, Eugene, and Szász, Otto. Some nonnegative trigonometric polynomials connected with a problem in probability. J. Research Nat. Bur. Standards 48, 139–146 (1952).

It results from previous work by the authors [Canadian J. Math. 3, 140–144 (1951), especially (4.4); these Rev. 12, 823] that the reciprocal of a polynomial whose roots are all single and have the same imaginary part is the Fourier transform of a distribution function if and only if (1) the polynomial has one purely imaginary root ai ($a\neq0$) and n pairs of complex roots

$$\pm b_j+ai \quad (0<b_1<\cdots<b_n,\ j=1, \cdots, n),$$

and (2) the determinant

$$\begin{vmatrix} \sin^2 b_1t & \sin^2 b_2t & \cdots & \sin^2 b_nt \\ b_1^2 & b_2^2 & \cdots & b_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ b_1^{2n-2} & b_2^{2n-2} & \cdots & b_n^{2n-2} \end{vmatrix} \geqq 0 \quad (-\infty<t<\infty).$$

The authors now investigate certain cases in which the b_j's are integers. They show that (2) is satisfied (A) when $b_j=j$, or (B) when $b_j=2j-1$; and that (C) when the b_j's are the first $n+1$ positive integers with k omitted $(1\leqq k\leqq n)$ (2) holds if and only if $2k^2\geqq n+1$. They also settle the case (D) that is like (C) but with odd integers, and the cases (E) and (F) corresponding to (C) and (D) but with two integers omitted: the conditions in cases (D), (E), (F) are too elaborate to be reproduced here. In the course of their proofs they obtain certain linear relations between generalized Vandermonde determinants formed of odd integers.

H. P. Mulholland (Birmingham).

Spiegel, M. R. The Dirac delta-function and the summation of Fourier series. J. Appl. Phys. **23**, 906–909 (1952).

Cinquini, Silvio. Sopra il problema dell'approssimazione delle funzioni quasi-periodiche. Ann. Scuola Norm. Super. Pisa (3) **5**, 245–267 (1951).

Il s'agit d'apporter des précisions sur le mode de convergence vers la fonction $f(x)$ p.p. (presque périodique) S^μ $(\mu\geqq1)$ de sa suite de polynomes $\sigma_{B_p}(x)$ de Bochner-Féjer; l'auteur donne, par exemple, des conditions suffisantes pour que

$$\lim_{p\to\infty}\ \sup_{-\infty<\alpha<+\infty}\int_\alpha^{\alpha+1}|g(x)|\phi(|f-\sigma_{B_p}|)dx=0,$$

g étant une fonction semi-continue sommable dans tout intervalle fini, f une p.p. S, et ϕ une fonction convexe ($\phi(u)\geqq0$ pour $u\geqq0$, $\phi(0)=0$) (ces conditions, relatives à f et g, sont trop longues à énoncer pour être rapportées ici). Ainsi on peut prendre $\phi=u^m$ lorsque f est p.p. $S^{m\mu}$, lorsque g est sommable ainsi que $|g|^\nu$ dans tout intervalle fini, et que $\int_\alpha^{\alpha+1}|g|^\nu dx$ est borné indépendamment de α $(1/\mu+1/\nu=1,\ m\geqq1,\ \mu>1)$. L'auteur examine ensuite le cas des fonctions p.p. de Bohr et absolument continues; il donne un exemple d'une telle fonction dont la dérivée f' n'est pas p.p. S.

J. Favard (Paris).

Burkill, H. Cesàro-Perron almost periodic functions. Proc. London Math. Soc. (3) **2**, 150–174 (1952).

La distance de deux fonctions $f(x)$ et $g(x)$ $(-\infty<x<+\infty)$, intégrables au sens de Cesàro-Perron (CP) (définition de J. C. Burkhill), est définie par:

$$D_{CP}(f,g)=\sup_{\substack{-\infty<y<+\infty\\ 0\leqq h,\ k\leqq1}}\left|\int_y^{y+h}dx\int_x^{x+k}[f(t)-g(t)]dt\right|;$$

l'intégrale par rapport à t est prise au sens CP, celle par rapport à x est alors une intégrale de Denjoy. Cette distance jouit des propriétés habituelles. Une fonction intégrable CP, $f(x)$, est dite p.p. (presque périodique) au sens CP si, pour tout $\epsilon>0$, il y a un ensemble relativement dense de nombres τ tels que:

$$D_{CP}[f(x+\tau),f(x)]<\epsilon.$$

L'ensemble des fonctions CP p.p. est identique à la fermeture CP de l'ensemble des polynomes trigonométriques. Toute fonction CP p.p. a une moyenne de Cesàro du deuxième ordre définie par:

$$\lim_{X\to\infty}\frac{2}{X}\int_0^X\left(1-\frac{x}{X}\right)f(x)dx.$$

On déduit de ce fait l'existence d'une série de Fourier et le théorème d'unicité, puis un résultat sur l'ordre de grandeur des coefficients de Fourier, et un sur la sommabilité de séries particulières. Partant de l'intégrale symétrique CP (SCP), à partir d'une base (définition de J. C. Burkhill) on a des résultats analogues et, de plus, on obtient une généralisation d'un résultat de Denjoy sur les séries trigonométriques ordinaires; soit

$$f(x)=\frac{a_0}{2}+\sum_1^\infty(a_n\cos\lambda_nx+b_n\sin\lambda_nx)\quad(\lambda_1>0,\ \lambda_{n+1}-\lambda_n\geqq l>0)$$

une série convergente pour tout x, alors $f(x)$ est SCP intégrable, avec une base appropriée, elle est aussi SCP p.p., et la série du second membre est sa série de Fourier.

J. Favard (Paris).

Berezanskiĭ, Yu. M. On the theory of almost periodic functions relative to translations in hypercomplex systems. Doklady Akad. Nauk SSSR (N.S.) **85**, 9–12 (1952). (Russian)

Let Q be a locally compact metric space, on which is defined an integral $\int x(t)dt$ corresponding to a measure function m. Let (Q) be the system of Borel sets of Q with compact closure and let $C(A,B,t)$, $A,B\in(Q)$, $t\in Q$, be an absolutely additive set function of each of the variables A and B for fixed values of the other variables, continuous in t for fixed A and B, and satisfying $C(A,B,t)=C(B,A,t)$ and $\int C(A,B,r)d_r(E_r,D,t)=\int C(B,D,r)d_rC(A,E_r,t)$. The convolution $(x*y)(t)=\int x(r)d_r\int y(s)d_sC(E_r,E_s,t)$ is assumed continuous with respect to the norm $\|x\|=\int|x(t)|dt$. The functional space $L_1(Q)$ is then organized as a ring, which is called a hypercomplex system with basis Q. It is called normal if there exists a homeomorphism $t\to t^*$ in Q such that the set function $C(A,B,D)=\int_DC(A,B,t)dt$ satisfies $C(A,B,D)=C(D,B^*,A)$ and $m(A)=m(A^*)$. If $U_n(t)$ denotes the sphere in Q with center t and radius $1/n$, the operator

$$(T_rx)(s)=\lim_{n\to\infty}\frac{1}{m(U_n(r))m(U_n(s))}\int C(U_n(r),U_n(s),t)x(t)dt$$

is a generalized translation of the function $x(t)$ [cf. B. M. Levitan, Mat. Sbornik 16(58), 259–280 (1945); these Rev. 7, 254]. A function $f(t)$ is called almost periodic if the set of translated functions $(T_rf)(s)$, $r\in Q$, is compact with respect to uniform convergence. The functions $\chi(t)$ satisfying $(T_r\chi)(s)=\chi(r)\chi(s)$ are called characters. The mean value is a functional M satisfying $\underset{t}{M}((T_{t^*}x)(r)y(t))=\underset{t}{M}(x(t)(T_{t^*}y)(r))$ and $\underset{t}{M}(|(T_tx)(r)|)\ M(|x(t)|)$. With the inner product $(f,g)=M(f(t)\overline{g(t)})$ the characters satisfying $(\chi,\chi)>0$ will form an orthogonal system and every almost periodic function will satisfy Parseval's equation

$$(f,f)=(f_0,f_0)+\sum_\chi|(f,\chi)|^2/(\chi,\chi),$$

where f_0 is the projection of f on the subspace of functions $u(t)$ satisfying $\underset{r}{M}(|\underset{t}{M}((T_{t^*}g)(r)u)|^2)=0$ for every almost periodic g. A function $x(t)$ is called even if $x(t^*)=x(t)$. The even functions form a hypercomplex system on a basis consisting of the points (t,t^*) and its translation operator will be $(T_{r'}x)(s')=\frac{1}{2}((T_rx)(s)+(T_{r^*}x)(s))$, where $r'=(r,r^*)$; $s'=(s,s^*)$. This result is applied to almost periodic functions on a group. Details of the proofs are not given. [Cf. Berezanskii, same Doklady 81, 329–332, 493–496 (1951); these Rev. 13, 952, 953.]

H. Tornehave (Lyngby).

Head, J. W. The decomposition of functions. Proc. Cambridge Philos. Soc. **48**, 733–734 (1952).

If a function $f(t)$ behaves asymptotically as $t^n e^{-t}$ when the time scale is suitably chosen, $f(t)$ can be advantageously decomposed into a series of Laguerre functions, which are mutually orthogonal over the range 0 to ∞. The coefficients in this series are usually obtained by integrations which use these orthogonal properties. Here they are obtained in terms of the values of the various Laguerre functions and of $f(t)$ and $dL_n(t)/dt$ when the Laguerre function $L_n(t)$ has a zero.

E. Frank (Chicago, Ill.).

Civin, Paul. Multiplicative closure and the Walsh functions. Pacific J. Math. **2**, 291–295 (1952).

The author considers a system of functions $\{\lambda_n(x)\}$ on the interval $[0, 1]$, which forms a normalized orthogonal set in L^2 of that interval and is closed under multiplication. The author proves that such a system $\{\lambda_n(x)\}$ always forms a group and that, under a suitable assumption on the sets $P_n=\{x; \lambda_n(x)=1\}$, there is a measure-preserving transformation T of $[0, 1]$ such that $\nu_n(x)=\psi_n(Tx)$, where $\{\nu_n(x)\}$ is a suitable rearrangement of $\{\lambda_n(x)\}$ and $\{\psi_n(x)\}$ is the family of Walsh functions.

K. Iwasawa.

Levitan, B. M. Expansion in Fourier series and integrals with Bessel functions. Uspehi Matem. Nauk (N.S.) **6**, no. 2(42), 102–143 (1951). (Russian)

The formulae for expansion of functions in Fourier-Bessel series and Fourier-Hankel integrals are derived [cf. Titchmarsh, Eigenfunction expansions . . . , Oxford, 1946; these Rev. **8**, 458]. The generalised translation operator defined by $T_x^y f(x)=g(x, y)$ where

$$\frac{\partial^2 g}{\partial x^2}+\frac{2p+1}{x}\frac{\partial g}{\partial x}=\frac{\partial^2 g}{\partial y^2}+\frac{2p+1}{y}\frac{\partial g}{\partial y},$$

$$g(x, 0)=f(x),\quad g_y(x, 0)=0,\quad p>-\tfrac{1}{2},$$

is discussed; it is proved to define a self-adjoint operator in the space of functions with $\|f\|^2=\int_0^\infty|f|^2x^{2p+1}dx$ and its connection with operators of Poisson and Sonine is shown. An analogy to Bochner's Fourier-Stieltjes formula is proved for generalised positive definite functions f which are such that $\int_0^\infty\int_0^\infty T_x^y f(x)g(x)\overline{g(y)}dxdy\geqq 0$ for every continuous function g vanishing outside a finite interval.

J. L. B. Cooper (Cardiff).

Ivašev-Musatov, O. S. On Fourier-Stieltjes coefficients of singular functions. Doklady Akad. Nauk SSSR (N.S.) **82**, 9–11 (1952). (Russian)

There exists a function $F(x)$ of modulus 1 such that $\int_0^\infty F(x)e^{ixy}dx=O(g(|y|))$, where g is any monotonic continuous positive function satisfying (1) $h(y)=\int_0^y[g(x)]^2dx\to\infty$ as $y\to\infty$; (2) $g(x)\to 0$ as $x\to\infty$; and (3) for some $\theta>1$ and A, $g(x)<Ag(\theta x)$ for all x. There exists a function F defined, continuous, monotonic and singular on $(0, 2\pi)$ such that $\int_0^{2\pi}e^{ixy}dF(x)=O(g(|y|))$ if g is any monotonic differentiable positive function on $(0, 2\pi)$ satisfying (1) and (4) $xg^2(x)\to 0$, $x^{1+\epsilon}g^2(x)\to\infty$ as $x\to\infty$, for any $\epsilon>0$; (5) for any $\theta\geqq 1$ and all x

$$\frac{g(x)}{g(\theta x)}\left(\frac{h(\theta x)+1}{h(x)+1}\right)^{1/2}\leqq\theta.$$

J. L. B. Cooper (Cardiff).

Duffin, R. J. Some simple unitary transformations. Ann. of Math. (2) **55**, 531–537 (1952).

On the analogy of Fourier transforms in L_2 the author considers "unitary" transformations on the space S of complex-valued functions $f(x)$ defined on the real axis for which $\int_{-\infty}^{\infty}|f(x)|^2dx<\infty$. A "unitary" transformation U of S is defined by the properties: (a) if $f(x)\in S$, then $Uf(x)=g(x)\in S$; (b) $\int_{-\infty}^{\infty}|g(x)|^2dx=\int_{-\infty}^{\infty}|f(x)|^2dx$; (c) U has an inverse; (d) U is linear. The main theorem of the paper, which gives a formula for defining such transformations, and the proof of which is elementary, is as follows: Theorem. Let $p(x)$ be a continuous complex-valued function such that if $P(x)=\int_{-\infty}^{x}|p(t)|^2dt$, then $0<P(x)<\infty$ for all x, and $P(x)\to+\infty$ as $x\to+\infty$. Let $q(-x)=p(x)/P(x)$. Then the relation

$$g(-x)=f(x)-\bar{q}(-x)\int_{-\infty}^{x}p(t)f(t)dt$$

defines a unitary transformation $g(x)=Uf(x)$ with an inverse given by

$$f(-x)=g(x)-\bar{p}(-x)\int_{-\infty}^{\infty}q(t)g(t)dt.$$

K. Chandrasekharan (Bombay).

Marmarašvili, G. A. Frobenius' theorem for double integrals. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 393–400 (1948). (Russian)

Let $f(x, y)$ be Lebesgue integrable over each finite rectangle $0\leqq x\leqq a$, $0\leqq y\leqq b$, and let

$$\varphi(x, y)=\int_0^x\int_0^y f(u, v)dudv,\quad F(x, y)=\frac{1}{xy}\int_0^x\int_0^y\varphi(u, v)dudv.$$

If $\lim_{x,y\to\infty}F(x, y)=S$, and if there exist constants A and α and β for which $0<\alpha$, $\beta<2$ and $|\varphi(x, y)|\leqq Ax^\alpha y^\beta$, then

$$\Phi(x, y)=\int_0^\infty\int_0^\infty e^{-xu-yv}f(u, v)dudv$$

exists when $x, y>0$ as $\lim_{A,B\to\infty}\int_0^A\int_0^B$. The author states and purports to prove an invalid conclusion, namely, that $\Phi(x, y)$ converges restrictedly to S as $x, y\to\infty$. It seems that a modification of his work suffices to prove a correct conclusion, namely, that $\Phi(x, y)$ converges restrictedly to S as $x, y\to 0$. The main steps in the proof involve integrations by parts which show that, under the given hypotheses,

$$\Phi(x, y)=xy\int_0^\infty\int_0^\infty e^{-xu-yv}\varphi(u, v)dudv$$

and then

$$\Phi(x, y)=x^2y^2\int_0^\infty\int_0^\infty uve^{-xu-yv}F(u, v)dudv.$$

Examples are given. *R. P. Agnew* (Ithaca, N. Y.).

Polynomials, Polynomial Approximations

Neville, E. H. On restricted cubics. J. London Math. Soc. **27**, 359–362 (1952).

The author derives parametric representations for all real cubic polynomials $f(x)$ for which $|f(\pm 1)|=1$ and $|f(x)|\leqq 1$ for $-1\leqq x\leqq 1$. His methods are claimed to be more elementary and his results simpler than those of Verblunsky [same J. **22**, 120–124 (1947); these Rev. **9**, 338].

M. Marden (Milwaukee, Wis.).

Carlitz, L. Note on irreducibility of the Bernoulli and Euler polynomials. Duke Math. J. **19**, 475–481 (1952).

The generating functions $te^{xt}/(e^t-1)=\sum_{m=0}^{\infty}B_m(x)\cdot t^m/m!$ and $2e^{xt}/(e^t+1)=\sum_{m=0}^{\infty}E_m(x)\cdot t^m/m!$ define the Bernoulli

and Euler polynomials, respectively, and

$$(t/(e^t-1))^k=\sum_{m=0}^{\infty}B_m^{(k)}\cdot t^m/m!$$

defines the Bernoulli numbers of order k; these are polynomials of degree m in k. For m odd $\geqq 3$, $B_m(x)$ has the factors x, $(x-\frac{1}{2})$, $(x-1)$; for m even $E_m(x)$ has the factors x and $(x-1)$ and for m odd the factor $(x-\frac{1}{2})$. Let $m=\sum a_ip^i$, $r=\sum b_ip^i$ be the p-adic expansions of m and r; then $\binom{m}{r}$ is prime to p if and only if $a_i\geqq b_i$ for all i. From this and the Staudt-Clausen theorem (if $(p-1)\nmid r$, B_r is integral (mod p); if $(p-1)|r$, then $pB_r\equiv -1 \pmod p$) it follows that the polynomial

$$B_{m(p-1)}(x)=\sum_{r=0}^{m(p-1)}\binom{m(p-1)}{r}B_rx^{m(p-1)-r}$$

is an Eisenstein polynomial, therefore irreducible. Furthermore, for $m=2^r$, $1\leqq s\leqq m-1$, $\binom{m}{s}\equiv 0 \pmod 2$ and it follows that $2B_m$ is an Eisenstein polynomial. In a similar way it follows that B_m is irreducible if $m=k(p-1)p^r$, $r\geqq 0$, $1\leqq k<p$. Using a generalization of Eisenstein's test [cf. Perron, Algebra, v. 1, 2nd ed., de Gruyter, Berlin-Leipzig, 1932, p. 195] it follows that if $2m+1=k(p-1)+1$, $1\leqq k<p$, then $B_{2m+1}(x)/x(x-\frac{1}{2})(x-1)$ has an irreducible factor of degree $\geqq 2m+1-p$. For the Euler polynomials it follows from similar considerations and Kummer's congruence [cf. N. Nielsen, Traité élémentaire des nombres de Bernoulli, Gauthier-Villars, Paris, 1923, chapter 14] that if $p\equiv 3 \pmod 4$, $m=p^r$, $r\geqq 1$, then $E_m(x)/(x-\frac{1}{2})$ is the product of at most r irreducible factors; and if $p>3$, $m=2p$, then $E_{2p}(x)/x(x-1)$ has an irreducible factor of degree $\geqq p-1$. Using Lagrange's interpolation formula and the Staudt-Clausen theorem it also follows that the polynomial in x, $pB_{p-1}(x)/x$ is an Eisenstein polynomial, hence irreducible.

E. Grosswald (Philadelphia, Pa.).

Markovitch, D. Sur un procédé de déterminer le plus grand commun diviseur de deux polynômes. Bull. Soc. Math. Phys. Serbie **4**, nos. 1–2, 37–41 (1952). (Serbo-Croatian. French summary)

Obreškov, N. Generalization of Descartes' theorem on imaginary roots. Doklady Akad. Nauk SSSR (N.S.) **85**, 489–492 (1952). (Russian)

Let $a_0x^n+a_1x^{n-1}+\cdots+a_n=0$ with real coefficients have p roots $r_je^{i\varphi_j}$ such that $|\varphi_j|<\pi/(n+2-p)$, $j=1, 2, \cdots, p$. It is proved that in the sequence of coefficients of the polynomial the number of changes of sign is equal to $p+2k$, where k is a non-negative integer. Other results are deduced from this one.

A. W. Goodman (Lexington, Ky.).

Parodi, Maurice. Sur une méthode de détermination du domaine des zéros de certains polynomes récurrents. C. R. Acad. Sci. Paris **234**, 1123–1124 (1952).

The author studies the polynomials $P_n(x)$ which satisfy the recursion formula

$$f_1(n+1)P_n(x)+[\varphi(n)-x]P_{n-1}(x)+f_2(n)P_{n-2}(x)=0$$

with $P_0=1$, $P_{-1}=0$; $\varphi(n)$, $f_1(n)$, and $f_2(n)$ being real functions of n with $f_1(n)>0$, $f_2(n)>0$. The zeros of $P_n(x)$ are real because they are the characteristic values of a certain matrix M. By use of a suitable diagonal matrix H, M transforms into $M_1=H^{-1}MH=M_1'+M_2'$ where $M_1'=\|\varphi(n+1-j)\delta_{ij}\|$ and $M_2'=\|a_{ij}\|$ with $\delta_{ij}=0$, $i\neq j$; $\delta_{ii}=1$; $a_{ij}=0$ for $j\neq i\pm 1$, $a_{i,i-1}=g(n+2-i)$, $a_{i,i+1}=g(n+1-i)$, and $g(n)^2=f_1(n)f_2(n)$. If λ_k, $k=1, 2, \cdots, n$, are the characteristic values of M_2', then the zeros of $P_n(x)$ lie on the interval of the real axis containing the points $\varphi(j)+\lambda_k$ $(j, k=1, 2, \cdots, n)$. For example, if $g(n)=A=\text{const.}>0$, then

$$\lambda_k=-2A^{1/2}\cos(k\pi/(n+1)).$$

M. Marden (Milwaukee, Wis.).

Walsh, J. L. Note on the location of zeros of extremal polynomials in the non-euclidean plane. Acad. Serbe Sci. Publ. Inst. Math. **4**, 157–160 (1952).

The non-euclidean (hyperbolic) plane H: $|z|<1$ is a subregion of the plane of the complex variable z. In H a NE (non-euclidean) polynomial is defined as a function of the form (1) $\lambda\prod_1^n(z-\alpha_k)/(1-\bar{\alpha}_kz)$, $|\lambda|=1$, $|\alpha_k|<1$, and the degree of this function is defined as n. This function determines an n-to-one map of the NE plane H onto itself, has precisely n zeros in H, and has the modulus unity on C: $|z|=1$. Here the following theorem is proved: Let the closed set E lie interior to some circle $|z|<r$ (<1). Let n be given. Then there exists at least one extremal polynomial (1), namely a NE polynomial (1) whose maximum modulus on E is not greater than the maximum modulus on E of any other NE polynomial (1). If E contains at least n distinct points, all zeros of this extremal polynomial lie in the smallest NE convex set K containing E.

E. Frank.

Aruffo, Giulio. Un'osservazione sull'approssimazione di una funzione continua per mezzo di una successione di funzioni razionali. Boll. Un. Mat. Ital. (3) **7**, 44–47 (1952).

Soit $f(x_1, \cdots, x_n)$ une fonction continue dans $E_n(0\leqq x_h\leqq 1)$, $f^{[m]}$ son polynome d'approximation de Stieltjes d'ordre m; on pose: $f^{(m)}=f^{[m]}/1^{[m]}$ (où $1^{[m]}$ est le polynome d'approximation de l'unité); alors si f admet une dérivée $D_{i_1+\cdots+i_n}f$ (i_h dérivations par rapport à x_h) continue dans E_n, on a:

$$\lim_{m\to\infty}D_{i_1+\cdots+i_n}f^{(m)}=D_{i_1+\cdots+i_n}f$$

uniformément dans tout domaine intérieur à E_n.

J. Favard (Paris).

Šuvalova, È. Z. On overconvergence of a sequence of polynomials. Mat. Sbornik N.S. **31(73)**, 76–87 (1952). (Russian)

The author derives some relationships between best (Tschebysheff) polynomial approximation of functions analytic on a bounded continuum and the properties of overconvergence and analytic continuation. The principal theorem proved is as follows. Let f be analytic on a bounded continuum K, and $P_n(z)$ be the polynomials of best approximation to f on K, while E_n is the best approximation itself. Let $\limsup_{n\to\infty}(E_n)^{1/n}=\rho/R$, $\rho<R$ and $\limsup_{n\to\infty}(E_n)^{1/n}\neq\liminf_{n\to\infty}(E_n)^{1/n}$. Then there will exist a subsequence $P_{n_k}(z)$ which converges in the neighborhood of each regular point of f on C_R. Here C_R designates a Kreisbild under the mapping function $\phi(z)$, $\phi(\infty)=\infty$, which takes the exterior of the unit circle onto that region adjacent to K which contains the point at infinity.

P. Davis (Washington, D. C.).

Džrbašyan, M. M. On weighted best approximation on the real axis by polynomials. Doklady Akad. Nauk SSSR (N.S.) **84**, 1123–1126 (1952). (Russian)

Let $f(x)$ be a function continuous on $(-\infty, \infty)$, and such that $\lim e^{-p(x)}f(x)=0$ as $|x|\to\infty$; let $E_n[f; p(x)]$ denote the

lower bound of $\max_x e^{-p(x)}|f(x)-Q_n(x)|$ for polynomials $Q_n(x)$ of degree n. The author assumes that $p(x)$ is even, differentiable, and has $xp'(x)\uparrow+\infty$ as $x\uparrow+\infty$, and recalls that $E_n[f;p(x)]\to 0$ as $n\to\infty$ if and only if $\int^\infty x^{-2}p(x)dx=-\infty$. [He refers to S. Bernšteĭn, Extremal properties of polynomials . . . , Moscow-Leningrad, 1937; to his own papers, same Doklady **62**, 581–584 (1948); **66**, 1037–1040 (1949); **67**, 15–18 (1949); these Rev. **10**, 364; **11**, 94, 95; and to inaccessible papers by Šaginyan and by himself. Other relevant references are Mandelbrojt, Duke Math. J. **11**, 341–349 (1944); Bull. Amer. Math. Soc. **54**, 239–248 (1948); Rice Institute Pamphlet, Special Issue, Houston, 1951; these Rev. **5**, 257; **9**, 416; **13**, 540; Ahiezer and Babenko, same Doklady **57**, 315–318 (1947); these Rev. **9**, 141; Carleson, Proc. Amer. Math. Soc. **2**, 953–961 (1951); these Rev. **13**, 632; Bernšteĭn, same Doklady **77**, 549–552, 773–776 (1951); these Rev. **12**, 814; **13**, 26; Videnskiĭ, ibid. **84**, 421–424 (1952); these Rev. **14**, 154.] The object of this paper is to estimate the magnitude of $E_n[f;p(x)]$ under supplementary hypotheses on $f(x)$ and to give converse theorems. The following results are given, with proofs for (1) and (3). (1) If $f(x)$ is continuous and bounded, together with its derivatives of orders up to and including k, then for any $q>1$ and $c>0$, $E_n[f;c|x|^q]\leqq K(c,q,k)n^{-k(1-1/q)}$. (2) If $f(z)$ is regular and bounded in the strip $|y|\leqq d$, then $E_n[f;c|x|^q]\leqq K\exp(-dn^{1-1/q})$. (3) Let $x=g(y)$ be the inverse of $y=p(x)$; without loss of generality assume that, for $y\geqq 1$, $g(y)$ exists, increases, and $g(y)\geqq 1$. Then if $\int^\infty x^{-2}p(x)dx=-\infty$ and $f(x)$ satisfies

$$E_n[f;p(x)]\leqq K\left\{\int_1^n[g(y)]^{-1}dy\right\}^{-1-\epsilon},$$

then $f(x)$ and $f'(x)$ are continuous on every finite interval. Results analogous to (1) and (3) are stated for approximation on a half-line. *R. P. Boas, Jr.* (Evanston, Ill.).

Berman, D. L. Approximation by interpolation polynomials of functions satisfying a Lipschitz condition. Doklady Akad. Nauk SSSR (N.S.) **85**, 461–464 (1952). (Russian)

The author considers Bernstein's interpolation polynomials [Zapiski Harkiv. Mat. Tov. ta Ukraïn. Inst. Mat. Nauk (4) **5**, 49–57 (1932)], for nodes x_j which are the zeros of the Čebyšev polynomials $T_n(x)$ on $[-1,1]$. If $l_n(x)$ denote the Lagrange interpolation polynomials with the same nodes, and p is a fixed integer, the polynomials are

$$A_n[f;x]=\sum_{j=1}^{n}{}'f(x_j)\{l_j(x)+(-1)^{i-1}l_{2pi_j}(x)\},$$

where $2p(i_j-1)<j<2pi_j$ and $\sum'$ omits multiples of $2p$. It is known that $A_n[f;x]\to f(x)$ uniformly when $f(x)$ is continuous. The author now considers the case when $f(x)$ satisfies a Lipschitz condition of order α with constant K. If $\delta^{(\alpha)}(x)$ denotes the maximum of $|f(x)-A_n[f;x]|$ for all $f(x)$ in this class, he proves that, for $0<\alpha<1$,

$$\delta^{(\alpha)}(x)\leqq C_1(K,\alpha,p)n^{-\alpha}+C_2(K,\alpha,p)n^{-\alpha}|T_n(x)|,$$

while

$$\delta^{(1)}(x)\leqq C_3(K,p)n^{-1}+\{C_4(K,p)n^{-1}\log n\}|T_n(x)|,$$

where $C_1,\cdots,C_4$ depend only on the indicated arguments and are given by explicit expressions; the orders (in n) of these estimates are exact. *R. P. Boas, Jr.*

★Mergelyan, S. N. Nekotorye voprosy konstruktivnoĭ teorii funkciĭ. [Certain questions of the constructive theory of functions.] Trudy Mat. Inst. Steklov., v. 37. Izdat. Akad. Nauk SSSR, Moscow, 1951. 91 pp. 4.50 rubles.

The author gives a systematic account of some results on degree of approximation by polynomials; some of these have been announced previously and reviewed in detail [Doklady Akad. Nauk SSSR **61**, 981–983 (1948); **62**, 23–26, 163–166 (1948); **79**, 731–734 (1951); these Rev. **10**, 242, 243; **13**, 222; cf. also Acta Sci. Math. Szeged **12**, Pars A, 198–212 (1950); these Rev. **12**, 176]. Although the arrangement of the material differs considerably from that adopted in the cited papers, this review will be confined to results stated (and proved) in the present monograph apparently for the first time (at least as far as accessible publications are concerned). Denote by $\rho_n(f,D)$ the best approximation to $f(z)$ on D by polynomials of degree at most n, that is, the minimum with respect to all such polynomials $P_n(z)$ of $\max_D|f(z)-P_n(z)|$. When D is the segment $(0,1)$, $\rho_n(f,D)$ is denoted by $E_n(f)$.

Among the new results established are the following [Chapter 3]. For a (mildly restricted) region D, if $\rho_n(f,D)<cn^{-A}$, then for any positive ϵ the $[\frac{1}{2}A-\epsilon]$th derivative of $f(z)$ is continuous in $\bar{D}$. There exist a Jordan region D and $f(z)$ analytic in D and continuous in $\bar{D}$ such that $\int_a^z f(z)dz\equiv F(z)\to\infty$ as z approaches each point of a set dense on the boundary. However, if $\sum\rho_n(f,D)<\infty$, then $F(z)$ must be continuous in $\bar{D}$. There exist a Jordan region D and a sequence of polynomials $\{P_n(z)\}$ converging uniformly to zero in $\bar{D}$, for which $\max|\int_a^z P_n(z)dz|\to\infty$.

Chapter 7 discusses quasi-analytic classes. The class Q_ϕ is the class of functions $f(x)$ on $(0,1)$ for which $\liminf E_n(f)/\phi(n)<\infty$. For suitable ϕ, if $\psi(\delta)\to 0$ rapidly enough as $\delta\to 0$ and f_1, f_2 belong to Q_ϕ and satisfy $|f_1(x)-f_2(x)|<\psi(|x-x_0|)$, $0<x_0<1$, then $f_1(x)\equiv f_2(x)$. Similarly, the class $U_\phi(F)$ is the class of functions continuous on F (a closed bounded point set not separating the plane) for which $\liminf\rho_n(F,f)/\phi(n)<\infty$, where $\phi(\infty)=0$. If M is any infinite subset of F, there corresponds to it a function $\phi(n)$ such that $U_\phi(M)$ has the quasi-analytic property: $f_1(z)$ and $f_2(z)$ of $U_\phi(M)$, coinciding on M, coincide on F. The author defines a Bernstein quasi-analytic class as follows: $f(x)$ belongs to $B\{n_k\}$ if $\lim_{k\to\infty}|\log E_{n_k}(f)|/n_k>0$. For $B\{n_k\}$ to be contained in a Denjoy-Carleman quasi-analytic class, it is necessary and sufficient that for every $C>1$ the series

$$\sum_{k=1}^{\infty}\min_{p\geqq 1}p^{-1}C^{\phi(p)/n}$$

diverges, where $\phi(n)=n_k$ for $n_k\leqq n<n_{k+1}$. Let $f(z)$ be analytic in $|z|<1$, continuous in the closed circle, and not identically zero, but with zeros at the points of a set M having $z=1$ as limit point. Let $\lambda(\psi)$ be defined, if $e^{i\psi}$ is not in M, as the maximum distance from $e^{i\phi}$ to M for $|\phi|\leqq\psi$; if $e^{i\psi}$ is in M, define $\lambda(\psi)$ by continuity. Then $\int_0^c\log\omega(\lambda(\psi))d\psi>-\infty$, where ω is the modulus of continuity of $f(z)$. If $\rho_n(f,|z|\leqq 1)<e^{-n/\log n}$, and $f(z)\not\equiv 0$, then $f(z)$ has only a finite number of zeros in $|z|\leqq 1$.

Chapter 8 is devoted to the Bieberbach polynomials $\pi_n(z)$ which minimize $\iint_D|P_n'(z)|^2dxdy$, with $P_n(0)=0$, $P_n'(0)=1$. Let D be a Jordan region whose boundary has a continuously turning tangent and let $w=f(z)$ map D on $|w|<r$, with $f(0)=0$, $f'(0)=1$. Then for $\epsilon>0$, the maximum on $\bar{D}$ of $|\pi_n(z)-f(z)|$ is $O(n^{-\frac{1}{2}+\epsilon})$, so that in particular

$\pi_n(z) \to f(z)$ uniformly on $\bar{D}$. This extends results of M. Keldyš [Mat. Sbornik N.S. 5(47), 391–401 (1939); these Rev. 2, 80].
R. P. Boas, Jr.

Geronimus, Ya. L. On some extremal problems in the space $L_\sigma^{(p)}$. Mat. Sbornik N.S. 31(73), 3–26 (1952). (Russian)

The author solves several extremal problems associated with the space $L_\sigma^{(p)}$, which is the linear space of functions $f(\xi)$ defined on the rectifiable Jordan curve $\xi=\xi(s)$ (s is arclength), with norm $\|f\|$ given by

$$\|f\|^p=(2\pi)^{-1}\int_C |f(\xi)|^p d\sigma(s),$$

where $\sigma(s)$ is a monotonic nondecreasing function with an infinite number of points of increase. On this space define a functional $A(f)=(2\pi i)^{-1}\int_C \xi^{-1}f(\xi)d\xi$. The author first shows that $\|A\|^q=(2\pi)^{-1}\int_C|\xi\{\sigma'(s)\}^{1/p}|^{-q}|d\xi|$, where $p>1$ and $q=p/(p-1)$, and he determines the functions for which $|A(f)|=\|A\|$ when $\|A\|$ is finite. By another result of the author [paper unavailable outside the USSR], the finiteness of $\|A\|$ is equivalent to the property that the system $\{\xi^{\pm k}\}_1^\infty$ is not closed in $L_\sigma^{(p)}$, that is, that linear combinations $\sum'^n_{k=-n}\alpha_k\xi^k$ are not dense in $L_\sigma^{(p)}$ ($\sum'$ omits $k=0$); this contrasts with the fact that $\{\xi^k\}_{-\infty}^\infty$ is closed in $L_\sigma^{(p)}$. In the special case where $p=2$ and C is the unit circle, the author obtains the following results.

Let $\sigma(s)=\tau(\theta)$, $2\pi c_k=\int_0^{2\pi}e^{-ik\theta}d\tau(\theta)$, so that $\Delta_m=|c_{i-k}|_0^m>0$. Let Δ'_{2n} be the minor of the central element of Δ_{2n}, and let $G_n(z)=\sum_{k=-n}^n\alpha_k z^k$. Then

$$\begin{aligned}(1)\quad M_n^2&=\{\min\|G_n\|/|A(G_n)|\}^2\\ &=\min_\alpha (2\pi)^{-1}\int_0^{2\pi}\left|1+\sum_{k=1}^n(\alpha_k e^{ik\theta}+\alpha_{-k}e^{-ik\theta})\right|^2 d\tau(\theta)\\ &=\Delta_{2n}/\Delta'_{2n},\end{aligned}$$

and

$$(2)\qquad \lim \Delta'_{2n}/\Delta_{2n}=(2\pi)^{-1}\int_0^{2\pi}\{\tau'(\theta)\}^{-1}d\theta\leqq\infty.$$

In addition, if $P_{2n}(z)=\alpha_0+\alpha_1 z+\cdots+\alpha_{2n}z^{2n}$ is the polynomial of degree $2n$ from the orthonormal system with weight $d\tau(\theta)$, then

$M_n^{-2}=|\alpha_{2n}|^2+|\alpha_{2n-1}|^2+\cdots+|\alpha_n|^2-|\alpha_{n-1}|^2-\cdots-|\alpha_0|^2.$

From (1) and (2) we then have a new property of orthogonal polynomials. As a further application the author gives the following theorem about general polynomials

$$K_s(z)=a_0+a_1z+\cdots+a_sz^s:$$

if $\nu=[s/2]$ and $|a_0|^2+\cdots+|a_{\nu-1}|^2>|a_\nu|^2+\cdots+|a_s|^2$, then $K_s(z)$ has at least one root in $|z|\geqq 1$; he remarks that a direct proof would be of interest.

Now let $w=\phi(z)=\alpha+z+b_2z^2+\cdots$ map the region B inside C on $|z|<1$, and let $z=\gamma(w)$ be its inverse function. Let G_0 be the functional

$$G_0(f)=\exp\left\{(2\pi)^{-1}\int_C\log|f(\xi)|\cdot|\gamma'(\xi)d\xi|\right\},$$

so that $G_0(f)$ is the geometric mean of $|f[\phi(e^{i\theta})]|$ on the unit circle. In this connection the author quotes a previous theorem of his [unavailable paper], that the set $\{\xi^k\}_0^\infty$ is closed in $L_\sigma^{(p)}$ if and only if

$$\int_C\log\sigma'(s)|\gamma'(\xi)d\xi|=\int_0^{2\pi}\log\tau'(\theta)d\theta=-\infty.$$

It was shown in special cases by Verblunsky [Proc. London Math. Soc. (2) 40, 290–320 (1935)], Szegö [Trans. Amer. Math. Soc. 37, 196–206 (1935)], the author [Zapiski Naučno-Issled. Inst. Mat. Meh. i Har'kov. Mat. Obšč. (4) 19, 35–120 (1948); these Rev. 12, 176] and Ahiezer [Lectures on the theory of approximation, Moscow-Leningrad, 1947; these Rev. 10, 33] that

$$\lim_{n\to\infty}\min_a\left\{(2\pi)^{-1}\int_0^{2\pi}\left|1+\sum_{k=1}^n a_ke^{ik\theta}\right|^p d\tau(\theta)\right\}=G(\tau')\equiv\exp\left\{(2\pi)^{-1}\int_0^{2\pi}\log\tau'(\theta)d\theta\right\};$$

the author establishes the general case. He proves that the norm of the functional G_0 is $\{G(\tau')\}^{-1/p}$, and determines the functions for which the norm is attained. In case $G(\tau')>0$, the author establishes the equivalence of three other conditions with this. Let $p_n(\xi)$ be the polynomial of degree n which minimizes $\|p_n\|$ under the condition $G_0(p_n)=1$. Let

$$D(z)=\exp\left\{-(2\pi p)^{-1}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\tau'(\theta)d\theta\right\},\quad |z|<1,$$

and let $\Delta(z)=D[\gamma(z)]$. One of the conditions equivalent to $G(\tau')>0$ is then that $\lim\|p_n-f_0\|=0$, where $f_0(\xi)=\Delta(\xi)/\Delta(\alpha)$, $\xi\in E$; $f_0(\xi)=0$ outside E, with E the set of points on C where $\sigma'(s)$ exists and is positive; a second condition is that $p_n(z)\sim\Delta(z)/\Delta(\alpha)$, uniformly in any region $|\gamma(z)|\leqq r<1$. A further result is that if $\sigma'(s)$ is constant, then $p_n(x)\sim\{\gamma'(x)\}^{1/p}$, $x\in B$, if and only if $\log|\phi'(z)|$ is represented by its Poisson integral in $|z|<1$.

Let D be the region outside C, let

$$w=\psi(z)=cz+c_0+c_1z^{-1}+\cdots,\quad c>0,$$

map D on the exterior of the unit circle; and let $z=\delta(w)$ be the inverse function. In the neighborhood of $w=\infty$ one can write $\delta^n(w)=f_n(w)+\lambda_n(w)$, where $f_n(w)=w^nc^{-n}+\cdots$ are the Faber polynomials of D. The author gives several conditions sufficient to ensure that (Γ) $\lambda_n(\xi)\to 0$ uniformly on C. Let $P_n(z)=z^n+\cdots$ deviate least from zero, in the $L_\sigma^{(p)}$ metric, among all polynomials of degree n, and let $h_{n,p}$ be the value of the deviation. Then if property Γ holds for C, the following conditions are equivalent: (I) $\int_C\log\sigma'(s)|\delta'(\xi)d\xi|>-\infty$; (II) $\lim c^{-n}h_{n,p}=\{G(\mu')\}^{1/p}>0$;

$$\text{(III)}\qquad \lim\|R_n-\phi^*\|=0,$$

where $R_n(x)=P_n(z)c^{-n}\delta^{-n}(z)$; $\phi^*(\xi)=\Delta_0(\xi)/\Delta_0(\infty)$ for $\xi\in E$ and 0 for ξ outside E; and, finally, with $d\sigma(s)=d\mu(\theta)$ and $D_0(z)$ defined like the reciprocal of $D(z)$ (with μ for τ), we have $\Delta_0(w)=D_0[\delta(w)]$; (IV) $P_n(z)\sim c^n\delta^n(z)\Delta_0(z)/\Delta_0(\infty)$ uniformly for $|\delta(z)|\geqq r>1$; special cases of the last formula are due to Szegö [loc. cit.] and Korovkin [Mat. Sbornik N.S. 9(51), 469–485 (1941); these Rev. 3, 114].
R. P. Boas, Jr. (Evanston, Ill.).

Russo, Salvatore. Sulla convergenza delle serie di polinomî di Legendre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 38, 168–175 (1952).

Using the formula of Schläfli (contour integral representation of $P_n(x)$) the author discusses in a straightforward manner the convergence of Legendre series $\sum a_nP_n(x)$, $-1\leqq x\leqq 1$, by establishing a relationship to the corresponding power series $\sum a_nz^n$ with appropriate z. This leads to various convergence theorems of general and special nature.
G. Szegö (Stanford, Calif.).

Delerue, Paul. Sur une généralisation à n variables des polynômes d'Abel-Laguerre. Ann. Soc. Sci. Bruxelles. Sér. I. 66, 13–20 (1952).

Using Laplace transforms (or what is the same, symbolic images) various properties of the generalized Laguerre polynomials $L_m^{\alpha_1\cdots\alpha_n}(x_1, x_2, \cdots, x_n)$ are proved. These polynomials can be defined by the generator series

$$\sum_{m=0}^{\infty}\frac{(m!)^{n-1}x_1^{\alpha_1}\cdots x_n^{\alpha_n}}{\Gamma(m+\alpha_1+1)\cdots\Gamma(m+\alpha_n+1)}L_m$$
$$=e^z\prod_{i=1}^{n}(x_i/z)^{\alpha_i/2}J_{\alpha_i}[2(zx_i)^{1/2}].$$

The recursion formula and the differential equation can be generalized. The special cases corresponding to Hermite polynomials deserve particular attention. The case $n=2$ is interesting. *G. Szegö* (Stanford, Calif.).

Dinghas, Alexander. Über einige Identitäten vom Bernsteinschen Typus. Norske Vid. Selsk. Forh., Trondheim **24** (1951), 96–97 (1952).

Generalizing the known identity

$$(1)\qquad \sum_{k=0}^{n} C_k^n x^k(1-x)^{n-k}(nx-k)^2=nx(1-x),$$

the author proves that

$$(2)\quad \sum C^q_{k_1,\cdots,k_n}x_1^{k_1}\cdots x_n^{k_n}(1-x_1-\cdots-x_n)^{k_{n+1}}\prod_{s=1}^{l}(qx_{r_s}-k_{r_s})^2$$
$$=\prod_{s=1}^{l}x_{r_s}\left[x_{r_s}\left(\frac{d}{dx}-q\right)^2+\frac{d}{dx}\right]x^q\Big|_{x=1},$$

where $q=k_1+\cdots+k_{n+1}$, $C^q_{k_1,\cdots,k_n}=q!/\prod k_n!$ and the $r_1, \cdots, r_l$ form a subset of $1, \cdots, n$. He states that (2) may be used to prove the convergence of a certain form of n-dimensional Bernstein polynomials to the generating function. [The reviewer remarks that (1) is also sufficient for this purpose, see G. G. Lorentz, Bernstein polynomials, University of Toronto Press, 1952, §2.9]. *G. G. Lorentz.*

Harmonic Functions, Potential Theory

Huber, Alfred. Über Wachstumseigenschaften gewisser Klassen von subharmonischen Funktionen. Comment. Math. Helv. **26**, 81–116 (1952).

This paper, divided into two parts, deals with entire subharmonic functions of finite order on n-dimensional Euclidean space. In part I functions of order <1 on the plane are studied by means of the Heins representation theorem [Ann. of Math. (2) **49**, 200–213 (1948); these Rev. **9**, 341]. A principal result is the extension to subharmonic functions of a theorem of Valiron-Wiman: if u is an entire subharmonic function of order $\rho<1$ and $m(r)=\inf u(z)$, $M(r)=\max u(z)$ for $|z|=r$, then

$$\limsup_{r\to\infty} m(r)/M(r)\geqq\cos\pi\rho.$$

Moreover, for $u\neq$const. and $\rho<\rho'<1$, upper density $\{r|m(r)>M(r)\cos\pi\rho'\}\geqq1-\rho/\rho'$, generalizing a theorem of Besicovitch for the analytic case. Refinements of the density theorem are given for $\rho<1/2$ and $\rho=0$.

In part II the characteristic constant $\alpha(\omega)$ of an open set ω is used in obtaining theorems of Phragmén-Lindelöf type. To define $\alpha(\omega)$, let ω^* be any subregion of ω with analytic boundary, and let λ_1^* be the smallest eigenvalue of $\Delta u+\lambda^*u=0$ on ω^* for $u(\neq0)$ vanishing continuously on the boundary of ω^* and twice continuously differentiable on ω^*; $\alpha(\omega)$ is then the non-negative solution of $x(x+n-2)=\lambda_1$, where $\lambda_1=\inf\lambda_1^*$ taken over all admitted ω^*. Theorem: For any region ω, $\alpha(\omega)=0$ if and only if the boundary of ω has capacity 0. Applications of α to Phragmén-Lindelöf theorems include a number of generalizations of theorems known for the analytic case or for $n=2$, but here established for a wider class of regions. E.g., hypothesis: 1) u a function subharmonic on a region Ω; 2) $\alpha(\Omega_r)\geqq\alpha_0(=\text{const.})>0$ for sufficiently large r, where $\Omega_r=\Omega\cap\{P|\,|P|<r\}$; 3) $\limsup_{P\to Q}u(P)\leqq0$ for Q any finite boundary point of Ω; 4) $\liminf_{r\to\infty}M(r)/r^{\alpha_0}\leqq0$; conclusion: $u\leqq0$ on Ω.

It should be noted that theorem 4 (part II) has been given in a somewhat less general form by Inoue [Mem. Fac. Sci., Kyūsyū Univ. A. **4**, 183–193 (1949); these Rev. **12**, 400]. Other results of Inoue on subharmonic Phragmén-Lindelöf theorems and their applications seem also to merit mention, for example, theorem 2 of J. Inst. Polytech. Osaka City Univ. Ser. A. **1**, 71–82 (1950); these Rev. **12**, 825. The reviewer also remarks that theorem 5 (part I) [for u an entire subharmonic function of order <1 on the plane, $M(r)=O(\log r)$ is equivalent to finiteness of the total mass of u] appears as a particular case of a theorem of the reviewer [Proc. Internat. Congress Math., Cambridge, 1950, v. I, Amer. Math. Soc., Providence, 1952, p. 374] characterizing functions of potential type. *M. G. Arsove.*

Royden, H. L. Harmonic functions on open Riemann surfaces. Trans. Amer. Math. Soc. **73**, 40–94 (1952).

The author is concerned with the decomposition of bounded real functions with a finite Dirichlet integral on open Riemann surfaces. A solution of this problem gives the solution of the first boundary value problem on regions with an ideal boundary and leads to a new approach to the classical type problem of Riemann surfaces. The paper is, in substance, the author's thesis [Harvard University, 1951].

The material covered is given by the titles of chapters: (I) Riemann surfaces: preliminary concepts; (II) Orthogonal projection and the space Γ; (III) The space BD on an open Riemann surface; (IV) The space BD on a bounded Riemann surface; (V) Some applications to the type problem. In the first two chapters, the foundations are laid for later reasonings. In particular, the apparatus of exterior differential calculus is derived ab ovo. On an open Riemann surface W, first order differentials $\alpha=adx+bdy$ are defined by complex functions a, b with the property that α is independent of the choice of the local uniformizer $z=x+iy$. The space Γ consists, by definition, of those differentials α for which the Dirichlet integral $D(\alpha)=\iint(|a|^2+|b|^2)dxdy$ is finite.

In chapter III, the space BD is defined to consist of bounded, piecewise continuously differentiable functions f on W whose differentials df belong to Γ. In BD, a topology is defined as follows: f_i is said to converge to f on BD if (1) $|f_i|$ are uniformly bounded, (2) $f_i\to f$ uniformly on every compact subregion and (3) $D(f-f_i)\to0$. Denote by K the subclass of BD consisting of those functions which vanish outside of a compact subregion. A function $f\in BD$ is said to be of class $\bar{K}$ if there is a sequence of functions φ_i in K such that $\varphi_i\to f$ in BD. The functions of the class $\bar{K}$ behave, in a certain sense, as though they "had zero boundary

values". The subclass of harmonic functions in BD is denoted by HBD. The main result is the following decomposition theorem: If $f\in BD$, then $f=f_K+u$, where $f_K\in \bar{K}$ and $u\in HBD$. The significance of the theorem lies in that it guarantees the solvability of the first boundary value problem, in the preceding sense, on an arbitrary abstract Riemann surface. The component f_K with "zero boundary values" can be expressed, in terms of f and the Green's function, by a simple integral formula. The parabolic type of a given surface can now be characterized by the condition $BD=\bar{K}$ or, which amounts to the same, $1\in\bar{K}$. Another necessary and sufficient condition for the parabolic type is the non-existence of a regular differential $\alpha\in\Gamma$ such that $d\alpha[1]\neq 0$, where $d\alpha$ is a linear functional defined on K by $d\alpha[\varphi]=-\int\int d\varphi\alpha$ and extended to $\bar{K}$ by continuity.

In the fourth chapter, the decomposition theorem is modified for Riemann surfaces V with a relative boundary R. Let O and N be the classes of functions on V which vanish or whose normal derivative vanishes on R respectively. Write $HK=HBD\cap\bar{K}$ and $HN=HBD\cap N$. Every function $f\in BD$ has now the orthogonal decomposition $f=f_0+u_k+u_N$ where $f_0\in\bar{K}O$, $u_K\in HK$ and $u_N\in HN$. This decomposition is unique up to an additive constant. The (relative) hyperbolic type of V is characterized by the existence of an $\alpha\in\Gamma$ such that $d\alpha[\varphi]=-\int\int d\varphi\alpha$ for all $\varphi\in\bar{K}$ and $d\alpha[1]\neq 0$. On a surface V with a compact relative boundary R, there is a norm-preserving isomorphism between the classes HO and HN, where the norm is defined as the supremum of the absolute values of the function. Moreover, if an open W is cut by a finite number of analytic Jordan curves into surfaces V with relative boundaries, then there are algebraic isomorphisms which map $HBD(W)$ onto $HO(V_1)+\cdots+HO(V_N)$ and $HN(V_1)+\cdots+HN(V_N)$ respectively. An application to the classification problem of Riemann surfaces reaffirms the relations

$$O_G\subseteq O_{HP}\subseteq O_{HB}\subseteq O_{HBD}=O_{HD}$$

[Royden, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **85** (1951); these Rev. **13**, 339].

In the closing chapter, the author gives a sufficient condition for the hyperbolic type in terms of a triangulation of the surface. The criterion is applied to covering surfaces whose branch points have a finite number of projections and to certain relatively non-ramified covering surfaces. It follows, in particular, that the Schottky covering surfaces of a closed surface are of hyperbolic type, a result originally due to Myrberg [ibid. no. **10** (1941); these Rev. **7**, 516].

L. Sario (Cambridge, Mass.).

Tsuji, Masatsugu. Existence of a potential function with a prescribed singularity on any Riemann surface. Tôhoku Math. J. (2) **4**, 54–68 (1952).

Let $\{F_n\}$ be an exhaustion of an open Riemann surface F and $g_n(z,\zeta)$ the Green's function of F_n with logarithmic pole ζ and Robin's constant γ_n. The author shows that a subsequence of $\{g_n-\gamma_n\}$ converges to a "generalized Green's function" $g(z,\zeta)$ of R. The proof is based on a number of estimates of g_n and γ_n on $F-\zeta$. No attempt is made to prove uniqueness of $g(z,\zeta)$. It is further shown that there exists, on every Riemann surface F, a function $g(z,\zeta_1,\zeta_2)$ with two logarithmic poles such that the Dirichlet integral $D(g)$ over a region not containing ζ_1,ζ_2 is finite. This is a special case of a result of the reviewer that there always exists a function p with $D(p-s)<\infty$ if the singular function s satisfies the condition $\int ds=0$ [C. R. Acad. Sci. Paris **229**, 1293–1295 (1949); Trans. Amer. Math. Soc. **72**, 281–295 (1952); these Rev. **11**, 342; **13**, 735]. Alternate proofs of the theorems of Osgood and Riemann-Roch are given. *L. Sario.*

Lewy, Hans. On the boundary behavior of minimal surfaces. Proc. Nat. Acad. Sci. U. S. A. **37**, 103–110 (1951).

This paper settles an outstanding problem in the theory of minimal surfaces and at the same time contains a method capable of many other applications. Let S be a minimal surface in the (ξ,ζ)-space bounded by a rectifiable Jordan curve C. Let C contain an analytic arc A. The author proves that S can be continued analytically across A, whereas thus far this has been known only in the case where A was a straight segment.

Assume that A contains the origin and is given by the equations: $\eta=\phi(\xi)$, $\zeta=\psi(\xi)$ with $\phi(0)=\psi(0)=\phi'(0)=\psi'(0)=0$. Map S conformally onto the upper half-plane of the z-plane in such a way that C is taken into the x-axis and the origin into the origin. Then S is represented parametrically by the relations $\xi=2\operatorname{Re}\lambda$, $\eta=2\operatorname{Re}\mu$, $\zeta=2\operatorname{Re}\nu$, where λ,μ,ν are analytic functions of $z=x+iy$, $y>0$, which are continuous for $y=0$. It must be shown that these functions are regular at $z=0$. Assume this to be known. Then there exists a function $\Xi(z)$ regular for small values of z, real for real z, and such that $\Xi(x)=\xi(x,0)$. This function will satisfy the differential equation

$$(*)\quad d\Xi=2[1+\phi'^2(\Xi)+\psi'^2(\Xi)]^{-1}[d\lambda+\phi'(\Xi)d\mu+\psi'(\Xi)d\nu].$$

Now the author proves that if $h>0$ is sufficiently small and q denotes the closed square with the corners h, $-h$, $h+2ih$, $-h+2ih$, then (1) the method of successive approximations gives a continuous solution of (*) in q under the initial condition $\Xi(0)=0$, (2) for $y=0$, Ξ coincides with $\xi(x,0)$, (3) $\Xi(z)$ can be continued analytically across the x-axis. (3) implies that λ,μ,ν can also be continued analytically across the x-axis. *L. Bers* (New York, N. Y.).

Lewy, Hans. A note on harmonic functions and a hydrodynamical application. Proc. Amer. Math. Soc. **3**, 111–113 (1952).

Let $U(x,y)$, $V(x,y)$ be conjugate harmonic functions defined for small values of $|x|$ and small negative values of y, and continuous, together with their derivatives, for $y=0$. Theorem: If on $y=0$ the functions U, V satisfy an analytic boundary condition of the form $U_y=A(x,U,V,U_x)$, then U and V can be continued analytically across $y=0$. The proof is based on the same idea as the author's proof of the analytic continuation theorem for minimal surfaces [see the preceding review]. Set $2F(x+iy)=U+iV$. A function $F_0(z)$, $y>0$, is constructed first for $x=0$, and then for a fixed x and variable y by solving ordinary differential equations involving the analytic function A. It turns out that $F_0(z)$ is analytic for $y>0$, continuous for $y=0$ and coincides with $F(z)$ on the x-axis.

As an application the author considers a steady incompressible two-dimensional flow of a heavy fluid with a free boundary, in the z-plane. If the kinetic energy per unit mass is positive, then z as a function of $\zeta=\phi+i\psi$ (where ϕ is the potential and ψ the stream-function) is analytic on the free streamline. *L. Bers* (New York, N. Y.).

Gerber, Robert. Sur une condition de prolongement analytique des fonctions harmoniques. C. R. Acad. Sci. Paris **233**, 1560–1562 (1951).

Let $\varphi=f+ig$ be an analytic function in a domain D, whose boundary includes an analytic arc γ. On γ the func-

tions f and g shall be defined and bounded, and g have a bounded derivative. A boundary condition of the form

$$\frac{dg}{ds}=\psi[f(s), g(s), s]$$

with an analytic ψ shall hold on γ. Then φ can be continued analytically across the interior points of γ. The proof indicated by the author proceeds in the following steps: Mapping D conformally on a circle, expressing φ in terms of g, proving that f has a derivative by means of Priwaloff's theorem, iteration of this argument to establish existence of all derivatives, construction of a majorant to show convergence of the Taylor series. [Reviewer's note: Essentially the same theorem has been given by H. Lewy; see the paper reviewed above.] *F. John* (New York, N. Y.).

Differential Equations

Viswanatham, B. The general uniqueness theorem and successive approximations. J. Indian Math. Soc. (N.S.) 16, 69–74 (1952).

Kamke a démontré le théorème général d'unicité suivant [Differentialgleichungen reeller Funktionen, Akademische Verlagsgesellschaft, Leipzig, 1930]: soit $\omega(x, z)$ continue et $\geqq 0$ pour $0<x\leqq a$, $z\geqq 0$, telle que 0 soit la seule intégrale de $z'=\omega(x, z)$ dans un intervalle $0<x<\alpha\leqq a$, ayant les propriétés $\lim_{x\to 0} z(x)=\lim_{x\to 0} z(x)/x=0$. Alors, si $f(x, y)$ est continue pour $0\leqq x\leqq \alpha$ et $|y|\leqq b$, et si on a dans cette région $|f(x, y_1)-f(x, y_2)|\leqq\omega(x, |y_1-y_2|)$, l'équation $y'=f(x, y)$ a une seule solution telle que $y(0)=0$. L'auteur prouve que si en outre on suppose que pour tout x, $\omega(x, z)$ est croissante en z, alors les approximations successives relatives à $y'=f(x, y)$ convergent vers l'unique solution de cette équation dans un voisinage de 0. Ce résultat améliore un théorème du rapporteur [Bull. Sci. Math. **69**, 62–72 (1945); ces Rev. **7**, 297] qui n'établissait cette convergence qu'en supposant en outre que $z'=\omega(x, z)$ n'a pas de solution satisfaisant à $\lim_{x\to 0} z(x)=\lim\inf_{x\to 0} z(x)/x=0$.
J. Dieudonné (Ann Arbor, Mich.).

Coddington, E. A., and Levinson, N. Uniqueness and the convergence of successive approximations. J. Indian Math. Soc. (N.S.) 16, 75–81 (1952).

Les auteurs annoncent avoir obtenu dès 1950 le résultat de B. Viswanatham [voir revue de l'article précédent]; ils en donnent une démonstration très analogue, en remplaçant l'hypothèse de continuité de $\omega(x, z)$ par l'hypothèse plus générale que $\omega(x, z)$ est mesurable et majorée par une fonction de x seul mesurable et intégrable dans tout intervalle ne contenant pas 0, la relation $z'=\omega(x, z)$ étant alors seulement vérifiée presque partout pour des fonctions z absolument continues. Dans la démonstration, les auteurs signalent que dans la première ligne de la p. 80, il faut supprimer le mot "first". *J. Dieudonné* (Ann Arbor, Mich.).

Rouquet la Garrigue, Victor. Le sens de l'étude qualitative des équations différentielles. Trabajos Estadística 2, 273–289 (1951). (French. Spanish summary)

An elementary exposition concerning the properties of the integral curves of differential equations of the form $y'=P(x, y)/Q(x, y)$ and $f(x, y, y')=0$. *W. Wasow.*

Sarantopoulos, Spyridon B. Some nuclei of contour integrals which satisfy linear differential equations. Bull. Soc. Math. Grèce 26, 109–130 (1952).

Solutions of linear differential equations can sometimes be expressed as contour integrals. The method is discussed, with examples, using Laplace's kernel: exp (zx), Laplace's kernel multiplied by a polynomial, and linear combinations of integrals like those obtained in the latter case.
T. E. Hull (Vancouver, B. C.).

Bruno, Angelo. Triangoli aritmetici ed equazioni differenziali in questioni di matematica finanziaria. Boll. Accad. Gioenia Sci. Nat. Catania (4) no. 5, 325–335 (1950).

Define $a_{r,s}$ by $a_{r,1}=a_{r,r}=1$ for $r=1, 2, 3, \cdots$ and

$$a_{r,s}=sa_{r-1,s}+a_{r-1,s-1}$$

for $s=2, 3, \cdots, (r-1)$ and $r=3, 4, \cdots$. The author considers the differential equation

$$\sum_{s=1}^{r} a_{r,s}x^s\frac{d^sy}{dx^s}=f(x), \quad f(x)=\sum_{v=1}^{n}(m+v)^r x^{m+v}$$

(m integer, $m\geqq 0$), and shows that this equation is satisfied by special types of annuities certain. The paper is based on some earlier results of G. Usai [Atti Accad. Gioenia Sci. Nat. Catania (5) **15**, no. 7' (1927)] which were not accessible to this reviewer. *E. Lukacs* (Washington, D. C.).

Štokalo, I. Z. On a generalization of the fundamental formula of the symbolic method. Ukrain. Mat. Žurnal 1, no. 3, 51–59 (1949). (Russian)

The linear system $x'-[A+\epsilon f(t)]x=Ce^{pt}$ is considered where A is a constant matrix, $f(t)$ is a matrix, C a constant vector, and p and ϵ parameters, the latter small. Formal solutions $x=\xi(t, p, \epsilon)e^{pt}$ are considered with $\xi=u_1/p+u_2/p^2+\cdots$ where u_j is $u_j(t, \epsilon)$. The form of u_j is readily ascertained. If Re p is different from the real parts of all the characteristic roots of A and if $f(t)$ is bounded, then the solution is shown to have more than formal significance. *N. Levinson.*

Butlewski, Z. Un théorème de l'oscillation. Ann. Soc. Polon. Math. 24 (1951), 95–110 (1952).

The author considers a system of differential equations $dx_i/dt=f_i(t, x_1, \cdots, x_n)$, $i=1, 2, \cdots, n$, where the functions f_i are continuous in the domain D: $t\geqq t_0$, $-\infty<x_i<\infty$. It is assumed, further, that through each point of D there passes one and only one solution curve of the system. A principal result is the following. If $x_{k-1}f_k>0$ when $x_{k-1}\neq 0$ $(k=1, 2, \cdots, n-1;\ x_0=x_n)$, $x_{n-1}f_n<0$ when $x_{n-1}\neq 0$; if the f_i have the property that $\bar{x}_j\geqq\bar{\bar{x}}_j$ $(j=n, 1, 2, \cdots, k)$ and $\bar{x}_j\leqq\bar{\bar{x}}_j$ $(j=k+1, \cdots, n-1)$ imply $f_k(t, \bar{x})\geqq f_k(t, \bar{\bar{x}})$; and, finally, if c_i are constants such that $c_{k-1}\int^\infty f_k(t, c)dt=+\infty$ when $c_{k-1}\neq 0$ $(c_0=c_n)$ and $c_{n-1}\int^\infty f_n(t, c)dt=-\infty$ when $c_{n-1}\neq 0$; then a solution $\{\varphi_i(t)\}$ such that $\varphi_k(t_0)\geqq 0$, $\varphi_n(t_0)\leqq 0$ is oscillatory for $t\geqq t_0$, the zeros of the functions $\varphi_i(t)$ are simple, and between two consecutive zeros of $\varphi_p(t)$ there is precisely one zero of $\varphi_q(t)$ $(p, q=1, 2, \cdots, n;\ p\neq q)$. (A solution $\{\varphi_i(t)\}$ is defined to be oscillatory for $t\geqq t_0$, if each function $\varphi_i(t)$ vanishes infinitely often on the interval $t\geqq t_0$.) This extends a result due to Fite [Trans. Amer. Math. Soc. **19**, 341–352 (1918)]. See also M. Zlamal [Časopis Pěst. Mat. Fys. **75**, 213–218 (1950); these Rev. **13**, 132]. *W. Leighton* (St. Louis, Mo.).

Malkin, I. G. On the stability of systems of automatic regulation. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 495–499 (1952). (Russian)

Consider the real system

$$(1)\qquad \dot x_s=\sum p_{sj}x_j+F_s(x_1,\cdots,x_n)$$

where: (a) the p_{sj} are constants such that the characteristic roots λ_j of their matrix all have negative real parts; (b) when every $|x_s|\leqq A$, then

$$(2)\qquad |F_s(x)|<Q\sum|x_j|$$

where Q is a constant. Let $x_{sj}(t,t_0)$ be a fundamental system of solutions of

$$(3)\qquad \dot x_s=\sum p_{sj}x_j$$

such that $\|x_{sj}(t_0,t_0)\|$ is the unit matrix of order n. If $\lambda_j=-\lambda_j'+i\lambda_j''$, let α be some positive number less than the least λ_j'. Then, for $t\geqq t_0\geqq 0$, $|x_{sj}(t,t_0)|<Me^{-\alpha(t-t_0)}$, where M is a positive constant. Let m be the largest number of terms in any one of the n linear expressions $\sum x_{sj}F_j(x)$. Theorem: If $Q<\alpha/Mm$, then the origin is asymptotically stable whatever the functions F_s satisfying (2) and whatever the initial values x_s^0 such that $|x_s^0|<\eta$ where

$$\eta<A(1/nM-mQ/n\alpha).$$

[Reference: M. A. Aĭserman, Avtomatika i Telemehanika **8**, 20–29 (1947); these Rev. **12**, 27]. *S. Lefschetz.*

Burton, Leonard P. Oscillation theorems for the solutions of linear, nonhomogeneous, second-order differential systems. Pacific J. Math. **2**, 281–289 (1952).

Consider the differential equation

$$(1)\qquad [K(x)y']'-G(x)y=A(x),$$

and the corresponding homogeneous equation

$$(2)\qquad [K(x)y']'-G(x)y=0,$$

where K, G, and A are continuous, $K>0$, $G<0$, and KG is non-increasing on an interval $a\leqq x\leqq b$. Let $u_1(x)$ and $u_2(x)$ be solutions of (2) such that $u_1(b)=0$ and $K(u_2u_1'-u_2'u_1)\equiv 1$. Finally, let $y_1(x)$ be any solution of (1). The author then shows that if either (i) $y_1(b)\leqq 0$, $A>0$ for $x>a$, and A/G is strictly decreasing with x, or (ii) $y_1(b)\geqq 0$, $A<0$, for $x>a$, and A/G is strictly increasing with x, then the zeros of $y_1(x)$ and $u_1(x)$ separate each other on $a\leqq x<b$. The proof is based on results due to W. M. Whyburn [Trans. Amer. Math. Soc. **30**, 630–640 (1928)]. *W. Leighton* (St. Louis, Mo.).

Barbuti, Ugo. Sopra un caso di "risonanza" per la equazione $x''+B(t)x=0$. Boll. Un. Mat. Ital. (3) **7**, 154–159 (1952).

The author presents a systematic technique for obtaining equations of the form $u''+(1+f(t))u=0$ where $f(t)\to 0$ as $t\to\infty$ which do not have all their solutions bounded as $t\to\infty$. His method is related to that given by Wintner [Amer. J. Math. **68**, 385–397 (1946); these Rev. **8**, 71].

R. Bellman (Santa Monica, Calif.).

Hartman, Philip, and Wintner, Aurel. An inequality for the amplitudes and areas in vibration diagrams of time-dependent frequency. Quart. Appl. Math. **10**, 175–176 (1952).

Let $\omega(t)$ be real and continuous. Let a and b be successive zeros of a solution $x(t)$ of $x''+\omega^2(t)x=0$. Then it is shown that

$$2\max_{a<t<b}x(t)<\int_a^b\omega^2(t)dt\int_a^b x(t)dt.$$

Moreover 2 is the best possible constant. *N. Levinson.*

Reeb, Georges. Remarques sur l'existence de mouvements périodiques de certains systèmes dynamiques. Arch. Math. **3**, 76–78 (1952).

The author considers a system:

$$(1)\qquad \begin{aligned}&\ddot x+f(x,\dot x)\dot x+x=g(x,\dot x,y_1,\cdots,y_n),\\ &\dot y_i=U_i(y_1,\cdots,y_n)+F_i(x,\dot x,y_1,\cdots,y_n),\end{aligned}$$

and states sufficient conditions for application of the Brouwer fixed point theorem to establish existence of a periodic solution. He points out the need for considering such systems in analyzing the single equation:

$$(2)\qquad \ddot x+f(x,\dot x)\dot x+x=g(x,\dot x,t),$$

in which g is periodic in t, inasmuch as (2) is often obtained in a physical problem as an approximation to (1).

W. Kaplan (Ann Arbor, Mich.).

Lyra, Gerhard. Über eine Konvergenzfrage bei der Auflösung linearer Differentialgleichungen in der Umgebung einer Stelle der Bestimmtheit. J. Reine Angew. Math. **189**, 238–242 (1952).

The author gives an improved treatment of the regular singular point for the differential equation

$$z^2w''+zP(z)w'+Q(z)w=0.$$

W. Leighton (St. Louis, Mo.).

Sestini, Giorgio. Criterio di stabilità in un problema di meccanica non lineare. Rivista Mat. Univ. Parma **2**, 303–314 (1951).

The main result concerns the equation $\ddot x+\phi(\dot x)+\omega^2x=f(t)$, with $\dot x\phi(\dot x)\geqq 0$, $|\phi(\dot x)|\to\infty$ as $|\dot x|\to\infty$, $f(t)$ BV in (t_0,∞). It is shown that any solution $x(t)$ which exists in (t_0,∞) is "stable", i.e., has $|x(t)|$ and $|\dot x(t)|$ bounded in (t_0,∞). ~~The author is apparently unaware that N. Levinson [J. Math. Physics 22, 181–187 (1943); these Rev. 5, 183] has proved more: if $\dot x\phi(\dot x)\geqq 0$ for large $|\dot x|$, $|\phi(\dot x)|\to\infty$ as $|\dot x|\to\infty$, and $f(t)$ is bounded in (t_0,∞), then every solution does exist in (t_0,∞) and is stable.~~ *G. E. H. Reuter* (Manchester).

Minorsky, N. Stationary solutions of certain nonlinear differential equations. J. Franklin Inst. **254**, 21–42 (1952).

Part I concerns the equation

$$(1)\qquad \ddot x+b\dot x+x+(a-cx^2)x\cos 2t+ex^3=0,$$

where $a=A\epsilon,\cdots,e=B\epsilon$ (ϵ small). Stable solutions of period 2π are found by a perturbation method, essentially that of van der Pol, presented (heuristically) in a manner similar to the rigorous treatment by Cartwright [Proc. Cambridge Philos. Soc. **45**, 495–501 (1949); these Rev. **11**, 249]. Results are then specialised by setting one or more parameters equal to zero. Part II contains results announced earlier [C. R. Acad. Sci. Paris **233**, 728–729 (1951); these Rev. **13**, 462].

G. E. H. Reuter (Manchester).

Staržinskiĭ, V. M. On the stability of unsteady motion in one case. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 500–504 (1952). (Russian)

The author studies the stability of

$$(1)\qquad \dddot x+p\ddot x+q\dot x+rx=0$$

where $p(t)$, $q(t)$, $r(t)$ are between two positive limits l and L, m and M, n and N. He replaces (1) by

$$(2)\qquad \dot x=y,\quad \dot y=z,\quad \dot z=-rx-qy-pz$$

and discusses the stability of (2) by the second Lyapunov method and a positive definite quadratic form.

S. Lefschetz (Princeton, N. J.).

Cohen, Hirsh. The stability equation with periodic coefficients. Quart. Appl. Math. 10, 266–270 (1952).

The author points out that some non-linear differential equations involving periodic phenomena may be studied as to the stability of these phenomena by deriving a subsidiary linear homogeneous differential equation with periodic coefficients. Instead of the usual Hill equation in this case, the author makes use of a slightly different linear homogeneous differential equation involving periodic coefficients. He sets out to discover if periodic solutions of this equation exist. Inserting periodic expressions into the equation he obtains a relation between the first few coefficients of the Fourier series as a condition for periodicity. Then he uses a power series solution in terms of a parameter of the differential equation. Combining this with the previous expressions he obtains conditions for periodicity and stability.

M. J. O. Strutt (Zürich).

Raymond, François-Henri. Sur la stabilité d'un asservissement linéaire multiple. C. R. Acad. Sci. Paris **235**, 508–510 (1952).

The author considers linear feedback systems having n input signals and n output signals, and gives what he states to be a sufficient condition for the stability of such a system. However, it appears to the reviewer, from the brief and not altogether clear sketch of the argument, that the condition is really a necessary, rather than a sufficient, condition. The condition, which is expressed in terms of properties of certain matrices, does not lend itself to a concise description.

L. A. MacColl (New York, N. Y.).

Roberson, Robert E. Synthesis of a nonlinear dynamic vibration absorber. J. Franklin Inst. **254**, 205–220 (1952).

A vibration absorber is a vibrating system that is coupled with another vibrating system in order to diminish the amplitude of the latter. The author shows in a simple case that with a proper choice of a non-linear driving force in the absorber the purpose of the attachment will be served for a wider frequency range than is possible with a linear absorber. Mathematically this involves the calculation, by Duffing's method, of the first non-linear perturbation term in a system of two second order differential equations.

W. Wasow (Los Angeles, Calif.).

Wittich, Hans. Über das Anwachsen der Lösungen linearer Differentialgleichungen. Math. Ann. **124**, 277–288 (1952).

Let $a_0(z), \cdots, a_{m-1}(z)$ be polynomials and let $w_1(z), \cdots, w_m(z)$ be a fundamental system of solutions of the linear differential equation

$$(*)\qquad w^{(m)}+a_{m-1}(z)w^{(m-1)}+\cdots+a_1(z)w'+a_0(z)w=0.$$

The author investigates the growth of the entire functions $w_1(z), \cdots, w_m(z)$ in the case in which all solutions of (*) are transcendental. A typical result is the following: If all solutions of (*) are transcendental, then for any fundamental system $w_1(z), \cdots, w_m(z)$ with the orders of growth $\lambda_1, \cdots, \lambda_m$, we have $m \leqq \sum_{\mu=1}^{m} \lambda_\mu$, with equality if and only if (*) has constant coefficients. The second part of the paper is concerned with the value distribution of transcendental solutions of the Riccati equation corresponding to (*) in the case $m=2$.

Z. Nehari (St. Louis, Mo.).

Humblet, Jean. Sur la définition des niveaux virtuels des noyaux atomiques et l'établissement de la formule de dispersion. Mém. Soc. Roy. Sci. Liège (4) **12**, no. 4, 114 pp. (1952).

This memoir presents a new and more rigorous derivation of the Breit-Wigner formula for nuclear resonances [Physical Rev. (2) **49**, 519–531 (1936)]. The following is typical of the mathematical problems that are discussed by the author. The function $v(r)$ is unknown but vanishes identically for $r>r_0$, a fixed distance. Let $\phi(r)$ in C' be a solution vanishing for $r=0$ of the equation $\phi''+[k^2-v(r)]\phi=0$ where k is a fixed positive quantity. It is clear that for large values of r, $\phi(r)=I\sin kr+Se^{ikr}$. The physicist is interested in the ratio S/I because this quantity describes the "scattering" of an incoming plane wave by a nuclear potential $v(r)$. It is easy to show that because of the continuity of $\phi(r)$ and ϕ' at $r=r_0$, one must have

$$(*)\qquad \frac{S}{I}=-i\frac{\sin kr_0\phi'(r_0,k)-k\cos kr_0\phi(r_0,k)}{e^{ikr_0}[\phi'(r_0,k)-ik\phi(r_0,k)]}.$$

Now, experiments can give S/I as a function of k and from this an attempt is made to derive information about nuclear forces. It can be proved that the numerator and denominator of S/I is an analytic function of k; consequently, the author uses a well-known theorem of Mittag-Leffler about the expansion of a meromorphic function in terms of its poles to derive the following result:

$$\frac{S}{I}=kP(k)+k\sum_1^\infty\left(\frac{R_n}{k-k_n}+\frac{R_n^*}{k+k_n^*}\right)$$

where k_n are the roots of $\phi'(r_0,k)-ik\phi(r_0,k)=0$, and $P(k)$ is an entire function of k. This result can easily be identified with the Breit-Wigner formula for this case.

B. Friedman (New York, N. Y.).

Persen, Leif N. Über die Wronskische Determinante bei selbstadjungierten Differentialgleichungen. Norske Vid. Selsk. Forh., Trondheim **24** (1951), 12–15 (1952).

For the self-adjoint equation $[f(x)y']'+\lambda g(x)y=0$ the Wronskian takes the form $A/f(x)$, A constant. This fact is illustrated for the special cases of the polynomials of Hermite, Laguerre, Tchebychev, and Legendre.

G. Szegö (Stanford, Calif.).

Levitan, B. M. On the completeness of the squares of characteristic functions. Doklady Akad. Nauk SSSR (N.S.) **83**, 349–352 (1952). (Russian)

The Sturm-Liouville problem for $y''+(\lambda-q(x))y=0$, $q(x)$ real, $y'(0)-hy(0)=0$, $y'(\pi)+Hy(\pi)=0$, h and H real constants, is considered. Let $\omega(x,\lambda_n)$ be the eigenfunction where $\omega(0,\lambda)=1$ and $\omega'(0,\lambda)=h$. Theorem: If $\int_0^{\pi/2} f(x)\omega^2(x,\lambda_n)dx=0$ for all n and $f(x)\in\mathfrak{L}(0,\frac{1}{2}\pi)$, then $f(x)=0$. The convergence of the expansion of $f(x)$ in the series $\chi(x,\lambda_n)=\omega^2(x,\lambda_n)-\frac{1}{2}$ is also considered. Reference is made to Borg [Acta Math. **81**, 265–283 (1949); these Rev. **11**, 353] where the completeness of squares of characteristic functions has been considered.

N. Levinson (Cambridge, Mass.).

Mendès, Marcel. Sur des équations se ramenant à la forme canonique. C. R. Acad. Sci. Paris **235**, 408–409 (1952).

It is proved that the so-called generalized canonical equations introduced by Meffroy [Bull. Astr. **16**, 213–219 (1952); these Rev. **14**, 100] can, by means of a linear change of variables, be reduced to the ordinary canonical form. The

theorem is essentially a special case of the known possibility of reducing the Birkhoff-Pfaffian equations to canonical form. [Cf. Féraud, same C. R. **190**, 358–360 (1930).]

D. C. Lewis (Baltimore, Md.).

Rašajski, Borivoje. Fonctions caractéristiques dans la théorie géométrique de systèmes des équations aux derivées partielles du 1^er ordre. Bull. Soc. Math. Phys. Serbie **3**, nos. 3–4, 37–44 (1951). (Serbo-Croatian. French summary)

This paper contains some historical remarks on the geometric theory of partial differential equations and a geometrically motivated construction of a complete integral, from the characteristic manifolds, for a system of partial differential equations of first order in one unknown.

M. Golomb (Lafayette, Ind.).

Persidskiĭ, K. On a system of partial differential equations. Izvestiya Akad. Nauk Kazah. SSR. **60**, Ser. Mat. Meh. **3**, 21–31 (1949). (Russian. Kazak summary)

Using rather complicated conditions the author removes certain assumptions of analyticity and boundedness from a stability theorem by Liapounoff. *L. Gårding* (Lund).

Markus, Lawrence. Global integrals of $fZ_x+gZ_y=h$. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 311–332 (1952).

In this paper the author obtains various characterizations of systems (1) $\dot{x}=f(x,y)$, $\dot{y}=g(x,y)$ whose family of solutions is "parallel", i.e., topologically equivalent to a family of parallel lines filling the plane. In particular, the following theorems are established: If f and g are of class C', $f^2+g^2>0$, and (1) has a parallel family of solutions, then there exists an analytic curve which meets each solution of (1) precisely once; if furthermore f and g are of class $C^{(n)}$ (or analytic), then (1) has an analytic integral $F(x,y)$ of class $C^{(n)}$ (or analytic) with non-vanishing gradient. If f and g are of class C' in the whole plane, $f^2+g^2>0$, and (1) does not have a parallel family of solutions, then $xf+yg$ has at least four roots on every sufficiently larger circle with center at the origin. *W. Kaplan* (Ann Arbor, Mich.).

Conti, Roberto. Sul problema iniziale per i sistemi di equazioni alle derivate parziali della forma

$$z_x^{(i)}=f^{(i)}(x,y;z^{(1)},\cdots,z^{(k)};z_y^{(i)}).$$

I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 61–65, 151–155 (1952).

The problem is to solve $z_x^\alpha(x,y)=f^\alpha(x,y;z^\beta(x,y);z_y^\gamma(x,y))$ (indices $\alpha,\beta,\gamma=1,\cdots,k$), with $x\geqq0$, subject to $z^\alpha(0,y)=\omega^\alpha(y)$ on an interval of the y-axis. Discussion indicates the desirability of requiring that all roots of the matrix f_s^α with $s=z_y^\gamma$ be real (hyperbolic case); hence, and for other stated reasons, set $\gamma=\alpha$. Suppose that the data $\omega^\alpha(y)$ are defined for $-\infty<y<\infty$, that $\ddot{\omega}^\alpha=d^2\omega^\alpha/dy^2$ are continuous, and that $|\dot{\omega}^\alpha|\leqq N$ and $|\ddot{\omega}^\alpha|\leqq M$. It is shown that a solution exists for $0\leqq x\leqq a$, $-\infty<y<\infty$ under conditions on $f^\alpha(x,y;z^\beta;q^\alpha)$ that include the following: for $0\leqq x\leqq a$ and all y, z^β, and q^α, the functions f_y^α, $f_{z^\beta}^\alpha$, $f_{q^\alpha}^\alpha$ do not exceed H (const.) in absolute value and are Lipschitzian in y, z^β, and q^α (const. L); moreover, with $K=\max(H,L)$, $2kK(1+a+kN+M)^2\leqq1$. The proof applies a method of E. Baiada [Ann. Scuola Norm. Super. Pisa (2) **12** (1943), 135–145 (1947); these Rev. **9**, 354], in which the solution is approximated by functions satisfying the initial conditions and satisfying the differential equation on the lines $x=ia/n$ $(i=0,1,\cdots,n-1)$.

F. A. Ficken (Knoxville, Tenn.).

Kamynin, L. I. The difference in uniqueness theorems for the heat conduction equation and for systems of difference-differential equations. Doklady Akad. Nauk SSSR (N.S.) **82**, 13–16 (1952). (Russian)

A. N. Tihonov [Mat. Sbornik **42**, 199–215 (1935)] showed that if $u(x,t)$ is a solution of the heat equation

$$(*)\qquad \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\quad t\geqq0,\quad -\infty<x<+\infty,$$

and if u satisfies the growth condition (for some positive constants c and t_0)

$$\max_{0\leqq t\leqq t_0}|u(x,t)|e^{-cx^2}\to0,\quad\text{as}\quad |x|\to\infty,$$

then u is uniquely determined by its values $u(x,0)$ on the x-axis. In the solution of the heat equation (*) by the method of finite differences in x, one is led to consider the infinite system of ordinary differential equations

$$(**)\qquad \frac{du_n}{dt}=u_{n+1}-2u_n+u_{n-1},\quad n=\cdots,-2,-1,0,1,2,\cdots.$$

The author shows, as a special case of a uniqueness theorem for a more general infinite system of ordinary differential equations

$$\frac{du_n}{dt}=f_n(t;u_{n-r},\cdots,u_{n+r}),\quad n=\cdots,-2,-1,0,1,2,\cdots,$$

where r is a fixed positive integer, that for the non-uniqueness of the solution of (**) one requires less restrictions on the growth of $u(x,t)$ than for the non-uniqueness of the solution of (*). *J. B. Diaz* (College Park, Md.).

Hirschman, I. I., Jr. A note on the heat equation. Duke Math. J. **19**, 487–492 (1952).

Let $u(x,t)$ satisfy the heat equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\quad(-\infty<x<\infty,\ -\infty<t\leqq c).$$

Appel [J. Math. Pures Appl. (4) **8**, 187–216 (1892)] has proved that if $0\leqq u(x,t)\leqq M$ $(-\infty<x<\infty,\ -\infty<t\leqq c)$, where M is a constant independent of x and t, then $u(x,t)$ must be constant. The principal theorem of the present paper shows that u is constant under weaker assumptions. This is stated as follows. Theorem: Let $(\xi,t)\in T$, where T is the set of points (ξ,t) with ξ in m-dimensional space and $-\infty<t\leqq c$. If (1) $u(\xi,t)\geqq0$ for $(\xi,t)\in T$, (2) u, u_t, u_{x_i}, $u_{x_ix_j}$ are all continuous for $(\xi,t)\in T$, (3) $\sum_{i=1}^m\partial^2u/\partial x_i^2=\partial u/\partial t$, $(\xi,t)\in T$, (4) $\liminf_{r\to\infty}r^{-1}\log M(r)\leqq0$, where $r=|\xi|$ and $M(r)\geqq\sup_{|\xi|\leqq r}u(\xi,c)$, then $u(\xi,t)$ is identically constant. This theorem is proved for $m=3$. Additional theorems giving integral representations of $u(\xi,t)$ under certain specified conditions are also included. *C. G. Maple.*

Nagumo, Mitio, et Simoda, Seturo. Note sur l'inégalité différentielle concernant les équations du type parabolique. Proc. Japan Acad. **27**, 536–539 (1951).

Westphal [Math. Z. **51**, 690–695 (1949); these Rev. **11**, 252] has stated conditions under which the inequalities

$$v_t\leqq f(x,t,v,v_{x_i},v_{x_ix_j})\quad\text{and}\quad w_t\geqq f(x,t,w,w_{x_i},w_{x_ix_j})$$

imply $v(x,t)<w(x,t)$ (where $x=(x_1,\cdots,x_n)$). The authors prove a much more general theorem where the same conclusion follows from an inequality

$$F(x,t,v,v_t,v_{x_i},v_{x_ix_j})>F(x,t,w,w_t,w_{x_i},w_{x_ix_j}).$$

The precise statement is too involved for a short review.

W. Feller (Princeton, N. J.).

Kampé de Fériet, Joseph. Sur une classe de solutions de l'équation de la chaleur. C. R. Acad. Sci. Paris **234**, 2139–2140 (1952).

A function $f(x)$ defined for $-\infty<x<\infty$ belongs to Wiener's class S if it is square integrable in each finite interval, and if the autocorrelation $\rho(h|f)$ exists for each h and is continuous for $h=0$. In this case $\rho(h|f)$ is the Fourier-Stieltjes transform of the spectrum $S(\lambda|f)$. The author states without proof that if $f\in S$ then the formal classical solution $u(t, x)$ of the heat equation is a solution such that $u(t, x)\in S$ for fixed t and $u(t, x)\to f(x)$ for almost all t. The autocorrelation $\rho(h|u)$ satisfies the heat equation $\rho_t=2\rho_{hh}$. Furthermore the author calculates the spectrum of u in terms of the spectrum of f and gives conditions that the former be absolutely continuous, etc. *W. Feller.*

Barenblatt, G. I., and Levitan, B. M. On some boundary problems for the equations of turbulent heat transfer. Izvestiya Akad. Nauk SSSR. Ser. Mat. **16**, 253–280 (1952). (Russian)

The equation of turbulent heat transfer

$$(1)\qquad \frac{\partial}{\partial z}\left[A(z)\frac{\partial R}{\partial z}\right]=u(z)\frac{\partial R}{\partial t}$$

may be put in the form

$$(2)\qquad \partial^2R/\partial x^2=q(x)\partial R/\partial t,\quad q(x)=u(z)A(z)$$

by use of the transformation (3) $x=\int_0^z[A(z)]^{-1}dz$. If then

$$(4)\qquad R(x, t)=y(x, \mu)e^{-\mu^{m+2}t},$$

the solution of equation (2) is made to depend upon the ordinary differential equation (5) $y''+\mu^{m+2}q(x)y=0$. The authors assume that $q(x)$ satisfies the following conditions: (a) $q(x)\geqq 0$ and can vanish only for $x=0$; (b) $q(x)$ has continuous second derivatives everywhere except possibly for $x=0$; (c) near the origin $q(x)=x^m(1+p(x))$, $p(0)=0$, $m\geqq 0$; (d) $\int_0^\infty(q(x))^{1/2}dx=\infty$;

$$(e)\qquad \int_0^\infty\left|\frac{5}{16}\frac{q'^2(x)}{q^{5/2}(x)}-\frac{q''(x)}{4q^{3/2}(x)}\right|dx<\infty.$$

With these conditions on $q(x)$, it is shown that the spectrum of the operator $q^{-1}(x)d^2/dx^2$ is continuous. There is also given a generalization of the formula of Poisson which may be used to construct solutions of physical problems associated with equation (1) and certain specified boundary conditions. Examples are included to illustrate the case in which the boundary conditions are non-homogeneous and equation (2) is replaced by

$$\partial^2R/\partial x^2=q(x)\partial R/\partial t-Q(x, t).$$

C. G. Maple (Ames, Iowa).

Resch, Daniel. Temperature bounds on the infinite rod. Proc. Amer. Math. Soc. **3**, 632–634 (1952).

Let $u(x, t)$ be a non-negative solution of the heat equation for all x and positive t. An upper bound for the values of $u(x, t)/u(x_1, t_1)$ at any point in the region with $0<t<t_1$ is given. If a monotonically increasing sequence of such solutions converges at one point, it converges uniformly in a region. *J. L. B. Cooper* (Cardiff).

Crank, J. Simultaneous diffusion and reversible chemical reaction. Philos. Mag. (7) **43**, 811–826 (1952).

The problem considered is one of diffusion in which some of the diffusing substance is immobilized by chemical reaction as diffusion proceeds. The reaction is considered to be of first order and reversible, the forward reaction proceeding at a rate proportional to the concentration of the amount of solute free to diffuse, and the backward reaction at a rate proportional to the concentration of the immobilized solute. The differential equations that govern the diffusion process are solved by the method of Laplace transforms. The solutions are derived for the diffusion of a limited amount of solute into a plane sheet, a cylinder, and a sphere respectively. The calculated values of M_t/M_∞, where M_t is the total amount of solute present in the sheet, cylinder or sphere at time t, and M_∞ the corresponding quantity after infinite time, are tabulated for the case of an infinite amount of solute and for a certain range of values of certain parameters which have physical meanings. Certain characteristic features of the diffusion-with-reaction are shown graphically, and a number of general conclusions are drawn in regard to the nature of the diffusion-reaction and its dependence on the ratio of the rate of diffusion to the rate of reaction.

H. P. Thielman (Ames, Iowa).

Manfredi, Bianca. Su la risoluzione della equazioni alle derivate parziali, del second'ordine, lineari e a coefficienti costanti. Rivista Mat. Univ. Parma **3**, 91–95 (1952).

In an earlier paper [Boll. Un. Mat. Ital. (3) **4**, 381–390 (1949); these Rev. **11**, 520] the author developed an operational method for solving partial differential equations. Here he indicates how these methods can be applied to partial differential equations with constant coefficients $a_{ij}=a_{ji}$ of the form

$$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+2a_{13}u_x+2a_{23}u_y+a_{33}u=f(x, y),$$

in case the determinant $|a_{ij}|=0$. *F. G. Dressel.*

Moon, Parry, and Spencer, Domina Eberle. Separability conditions for the Laplace and Helmholtz equations. J. Franklin Inst. **253**, 585–600 (1952).

Necessary and sufficient conditions for simple and R-separability of the Helmholtz and Laplace equations in Euclidean n-space are given. (An equation is R-separable if substituting

$$\varphi=U^1(u^1)\cdot U^2(u^2)\cdots U^n(u^n)/R(u^1, u^2, \cdots, u^n)$$

for the solution permits separation into n ordinary differential equations; it is simply separable if R is a constant.) The results are tabulated along with those for cylindrical and rotational coordinate systems in Euclidean 3-space.

T. E. Hull (Vancouver, B. C.).

Moon, Parry, and Spencer, Domina Eberle. Separability in a class of coordinate systems. J. Franklin Inst. **254**, 227–242 (1952).

Earlier results [see the preceding review] are applied to cylindrical and rotational coordinate systems in Euclidean 3-space when the solutions do not depend on all 3 variables.

T. E. Hull (Vancouver, B. C.).

Moon, Parry, and Spencer, Domina Eberle. Theorems on separability in Riemannian n-space. Proc. Amer. Math. Soc. **3**, 635–642 (1952).

Contains essentially the results of their other papers [see the two preceding reviews] with additional theorems for the case when the solutions do not depend on all n variables.

T. E. Hull (Vancouver, B. C.).

Višik, M. I. On strongly elliptic systems of differential equations. Mat. Sbornik N.S. **29**(71), 615–676 (1951). (Russian)

Let L be a real linear differential operator of order $2m>0$ with sufficiently differentiable coefficients operating on functions of n real variables with values in an N-dimensional space S. Let its principal part be

$$A=(-1)^m\sum A^{(k_1,\cdots,k_{2m})}(x)\partial_{k_1}\cdots\partial_{k_{2m}}\quad(\partial_k=\partial/\partial x_k),$$

where the $A^{(\cdot\cdot)}$ are N by N matrices. The symmetric part C of A is defined as

$$C=(-1)^m\sum C^{(k_1,\cdots,k_{2m})}(x)\partial_{k_1}\cdots\partial_{k_{2m}}$$

where $C^{(\cdot\cdot)}=\frac{1}{2}(A^{(\cdot\cdot)}+\tilde{A}^{(\cdot\cdot)})$ ($\sim$ denotes transposition) is the symmetric part of the matrix $A^{(\cdot\cdot)}$. The operator L is called strongly elliptic in a region D if

$$C(\xi)=\sum C^{(k_1,\cdots,k_{2m})}(x)\xi_{k_1}\cdots\xi_{k_{2m}}$$

is positive definite in D when $\xi_1,\cdots,\xi_n$ are real and not all zero. The simplest example is the polyharmonic operator $\Delta^m=(-1)^m(\partial_1^2+\cdots+\partial_n^2)^m$ for which $C(\xi)=(\xi_1^2+\cdots)^m$. Let $\Omega^0(D)$ be the set of all infinitely differentiable functions vanishing outside compact subsets of D. It is shown by a Fourier transformation that if C has constant coefficients, then the following inequality between the Dirichlet integrals corresponding to C and Δ^m, namely

$$(1)\qquad [Cu,u]\geqq\mu[\Delta^m u,u]\quad(u\in\Omega^0(D)),$$

where $[u,v]=\int_D(u,v)dx$ ((u,v) scalar product in S), and μ is some fixed positive number, is characteristic for strongly elliptic equations. If C does not have constant coefficients, it is replaced by some formally self-adjoint operator

$$\tilde{C}u=(-1)^m\sum\partial_{i_1}\cdots\partial_{i_m}(B^{(i_1\cdots i_m;\,j_1\cdots j_m)}(x)\partial_{j_1}\cdots\partial_{j_m}u)$$

whose principal part is C and it is shown that if L is strongly elliptic in D, then every point y in D has a neighborhood V such that (1) is valid for C when $u\in\Omega^0(V)$. In the sequel only bounded regions D are considered for which the fundamental inequality (1) is valid with C replaced by $\tilde{C}$. In the metric $[\Delta^m u,u]$, $\Omega^0(D)$ is closed to a Hilbert space $H=H(D)$ consisting, roughly speaking, of functions with (weak) square integrable derivatives of order $\leqq m$, those of order $<m$ vanishing at the boundary of D. The following facts, well-known in the classical case $m=N=1$ are shown. 1) The two equations $Lu=f$, $L^*v=h$, where $u,v\in H$, f and h are square integrable in D, L^* is the formal adjoint of L, ($[Lu,v]=[u,L^*v]$), and the operators L and L^* are otherwise suitably defined, constitute a Fredholm pair. 2) If the measure of $V D$ is small enough, either of them has a unique solution in $H(V)$. 3) The operator L is bounded from below, has a discrete spectrum, and, if λ is not an eigenvalue, $(L-\lambda)^{-1}$ is completely continuous. [Reviewer's note. It follows from a result announced by the reviewer [see the following review] that if $\tilde{C}$ is uniformly strongly elliptic and the matrices $B^{(\cdot\cdot)}$ uniformly continuous in D, then (1) is valid in the modified form $[\tilde{C}u,u]\geqq\mu[\Delta^m u,u]-c[u,u]$ (c a constant, $u\in\Omega^0(D)$), from which 1) and 3) still can be deduced.] *L. Gårding* (Lund).

Gårding, Lars. Le problème de Dirichlet pour les équations aux dérivées partielles elliptiques linéaires dans des domaines bornés. C. R. Acad. Sci. Paris **233**, 1554–1556 (1951).

The author extends his previous solution of the Dirichlet problem of a homogeneous elliptic equation with constant coefficients [see same C. R. **230**, 1030–1032 (1950); these Rev. **11**, 521] to arbitrary linear elliptic equations of order $2m$ with variable coefficients of class C^∞. Let H denote the space of functions f of class C^∞ in a bounded domain D, which vanish outside of a compact subset of D. Let $\mathcal{H}$ be the Hilbert space obtained from H by completion with respect to the norm $[f,f]$ given by the sum of the integrals of the squares of the absolute values of the derivatives of f of order $\leqq m$. Given a function g with $[g,g]<\infty$, the Dirichlet problem consists in finding a solution u of the differential equation of class C^∞, for which $g-u$ is in $\mathcal{H}$. The author's first theorem establishes for special elliptic equations an equivalence between the norm $[f,f]$ and a norm based on the "Dirichlet integral" of f appropriate to the differential equation. This is used to prove the alternative for the existence and uniqueness of the solution of the Dirichlet problem for the general elliptic equation. (Remark: Similar results have been given recently by F. E. Browder [cf. the following review].) *F. John.*

Browder, Felix E. The Dirichlet problem for linear elliptic equations of arbitrary even order with variable coefficients. Proc. Nat. Acad. Sci. U. S. A. **38**, 230–235 (1952).

Let K be a linear differential operator of order $2m$ with variable coefficients of the form

$$K=\sum p_{i_1\cdots i_m k_1\cdots k_m}\frac{\partial^{2m}}{\partial x_{i_1}\cdots\partial x_{i_m}\partial x_{k_1}\cdots\partial x_{k_m}}+R$$

where R is of order $<2m$. For a bounded domain D the j-norm of a function f is defined by $[(f,f)_j]^{1/2}$, where $(f,f)_j$ is the integral over D of the sum of the squares of the jth derivatives of f. The Hilbert space H is defined as the completion with respect to the m-norm of the space C_c^m of functions of class C^m, which vanish outside a compact subset of D. The functions of H are called "vanishing with their derivatives of order less than m on the boundary of D". The Dirichlet problem consists in finding a solution u of $Ku=0$ of class C^{2m} in D, for which $u-g$ is in H, where g is of class C^m in D and has finite m-norm.

The analogue of the Dirichlet integral for the operator $K-R$ is the expression

$$(f,f)=\int_D\sum p_{i_1\cdots i_m k_1\cdots k_m}\frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}}\frac{\partial^m f}{\partial x_{k_1}\cdots\partial x_{k_m}}dx.$$

Ordinarily K is called elliptic if the characteristics are imaginary, i.e., if the form of order $2m$ with coefficients $p_{i_1\cdots k_m}$ is definite. One of the difficulties encountered in the theory of general elliptic equations is that this definition does not imply the definiteness of the integrand in (f,f). This motivates the additional assumption made in the present paper that there exists a constant c such that $(f,f)\geqq c(f,f)_m$ for all f in C_c^m. Theorem I states that under this assumption and suitable differentiability conditions for the coefficients of K the Dirichlet problem has a solution if the problem with $g=0$ has only the zero solution. [Essentially the same theorem had been given by Gårding [see the preceding review] for equations with coefficients in C^∞, which are elliptic in the ordinary sense, by proving the existence of a constant t_0 such that for all $t>t_0$ the norm $[(f,f)+t(f,f)_0]^{1/2}$ is equivalent to the norm $[(f,f)_0+(f,f)_1+\cdots+(f,f)_m]^{1/2}$ in H.]

The author's second theorem states that "weak solutions" of the equation $Ku=\psi$ as defined by Green's identity are "strict" solutions of this equation for ψ in C^1. His third theorem expresses the fact that the equation $Ku=\psi$ has a solution with zero boundary values if and only if ψ is orthogonal to all solutions of the homogeneous adjoint

equation with zero boundary values, and that the number of independent null solutions is finite and the same for the equation and its adjoint. *F. John* (New York, N. Y.).

Pini, Bruno. Sul problema di Dirichlet per le equazioni a derivate parziali del secondo ordine di tipo ellittico. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 11, 325–333 (1951).

Let A be a domain in the (x, y)-plane. The boundary ſA of A is a simple closed continuous curve $x=x(s)$, $y=y(s)$, $0\leqq s\leqq L$, with continuous curvature. Let

$$\mathfrak{L}(u)\equiv \sum_{0\leqq h+k\leqq 2} a_{hk}\frac{\partial^{h+k}u}{\partial x^h\partial y^k}$$

where the a_{ik} are functions of x and y which are continuous together with their derivatives of order $\leqq h+k$. The quadratic form $\sum_{h+k=2}a_{hk}\lambda_h\lambda_k$ is supposed to be positive; moreover, $a_{00}\leqq 0$. Let finally f and φ be given continuous functions with domain A and ſA respectively. Under these assumptions a new existence proof for the Dirichlet problem

$$(1)\qquad \mathfrak{L}(u)=f \text{ in } A, \quad u=\varphi \text{ on ſ}A$$

is given which is based on a method previously employed by Cimmino [Ann. Scuola Norm. Super. Pisa (2) 7, 73–96 (1938); Rend. Circ. Mat. Palermo 61, 177–221 (1937); Rend. Sem. Mat. Univ. Padova 11, 28–89 (1940); these Rev. 8, 270], and which may be outlined as follows: Let Σ be the Banach space of all couples $(f(x, y), \varphi(s))$ (in a suitable norm) where $f(x, y)$ is measurable and essentially bounded in A while $\varphi(s)$ is continuous, and let $\Sigma_{\mathfrak{L}}$ be the subset of Σ which consists of all couples $(\mathfrak{L}(v), v(x(s), y(s)))$ where v and its derivatives up to and including order 2 are continuous in A. The main part of the proof consists in showing that the linear set $\Sigma_{\mathfrak{L}}$ is total and, therefore, dense in Σ [cf. Banach, Théorie des opérations linéaires, Warsaw, 1932, p. 58]. Consequently, if (f, φ) is a given element of Σ, there exists a sequence $(\mathfrak{L}(v_n), v_n(x(s), y(s)))$ of elements of $\Sigma_{\mathfrak{L}}$ converging to (f, φ). It is then shown that the v_n converge to a function v satisfying (1). During the proof mean-value formulas which are characteristic for a solution of the homogeneous equation adjoint to (1) play a major role. The author states that an extension to more general cases will be treated in a forthcoming paper. *E. H. Rothe.*

Miranda, C. Sui sistemi di tipo ellittico di equazioni lineari a derivate parziali del primo ordine, in n variabili indipendenti. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) 3, 85–121 (1952).

Let T be a bounded domain in Euclidean real n-space of points $x=(x_1, \cdots, x_n)$ and ſT its (connected) boundary. Let f_i and a_{ik} $(i, k=1, 2, \cdots, n; n>2)$ be functions of x defined in T, and let the quadratic form with coefficients a_{ik} be positive definite in T. The paper treats the problem of finding functions u_{ik}, f_i satisfying

$$(1)\qquad \sum_{k=1}^{n}\frac{\partial u_{ik}}{\partial x_k}=\sum_{k=1}^{n}a_{ik}\frac{\partial v}{\partial x_k}+f_i, \quad u_{ii}=0, \quad u_{ik}=-u_{ki} \quad \text{on } T$$

$$(1a)\qquad v=\bar{v} \quad \text{on ſ}T$$

where $\bar{v}$ is a preassigned function on ſT.

If the a_{ik} and f_i are often enough differentiable, the relation

$$(2)\qquad \sum_i\frac{\partial}{\partial x_i}\left\{\sum_k a_{ik}\frac{\partial v}{\partial x_k}+f_i\right\}=0$$

represents the integrability condition for (1). In this case, therefore, the solution of the problem (1), (1a) reduces to solving first the linear elliptic boundary value problem (2), (1a), and then to solving the integrable system (1) in which the right members are known. In particular, this method applies if the a_{ik} and f_i are polynomials.

The main object of the paper, however, is the treatment of the problem (1), (1a) under less restrictive conditions: continuity of the a_{ik}, quadratic integrability for the f_i (together with an additional restriction concerning the "infinities" of the f_i), and a Hölder condition for $\bar{v}$. Under these conditions existence (and as regards v also uniqueness) for problem (1), (1a) (under suitable regularity conditions on T) is proved. Moreover, if the a_{ik} and f_i satisfy certain Hölder conditions, the u_{ik} and v will also satisfy Hölder conditions in the interior of T. Under suitable conditions on $\bar{v}$ these Hölder conditions will still hold on ſT.

In order to obtain these results, the a_{ik} and f_i are approximated by polynomials $a_{ik}{}^m$, $f_i{}^m$ $(m=1, 2, \cdots)$. Let v^m be the solution of the problem (2), (1a) with the a_{ik}, f_i replaced by $a_{ik}{}^m$, $f_i{}^m$ respectively. The essential point is then to establish the compactness of the sequence v^m. To do this, delicate new estimates concerning solutions and their first derivatives of the problem (2), (1a) are necessary, and the main part of the paper is devoted to the proof of these estimates.

In the case of $n=2$ independent variables, the system (1) had been treated previously by various authors [Morrey, Trans. Amer. Math. Soc. 43, 126–166 (1938); Lichtenstein, Bull. Internat. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. Sér. A. Sci. Math. 1916, 192–217; E. Hopf, Math. Z. 30, 404–413 (1929)]. *E. H. Rothe* (Ann Arbor, Mich.).

Drăganu, Mircea. Some formulas in problems with oblique derivatives. Acad. Repub. Pop. Române. Bull. Şti. Ser. Mat. Fiz. Chim. 2, 567–574 (1950). (Romanian. Russian and French summaries)

In order to treat "oblique derivative" boundary value problems, Picone and Miranda [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 29, 160–165 (1939)] and Tolotti [ibid. 29, 285–293 (1939)] developed a Green's identity

$$\iiint_D [vE(u)-uE'(v)]d\tau + \iint_S\left(a^{(l)}v\frac{du}{dl}-a_*{}^{(l)}u\frac{dv}{dl_*}+b^{(l)}uv\right)dS=0,$$

where $E(u)=\sum_1^r a^{ik}\partial^2u/\partial x_i\partial x_k+\sum_1^r b^i\partial u/\partial x_i+cu$, $a^{ik}=a^{ki}$, and E' is its formal Lagrange adjoint, D is a domain and S its boundary, and l is a continuously varying unit vector on S, never tangent to S at a regular point of S (here d/dl and d/dl_* take the place of d'Adhémar's "conormal derivative" in the usual form of Green's identity [cf. Hadamard, Le problème de Cauchy . . . , Hermann, Paris, 1932, p. 88]. The author applies Tolotti's formula to several two and three dimensional operators, e.g., $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$, and obtains explicit formulas (analogous to "Green's third identity" in the terminology of Kellogg [Foundations of potential theory, Springer, Berlin, 1929]). For example, for the operator $\Delta-\partial^2/\partial t^2$, a formula analogous to Kirchhoff's formula (which expresses Huyghens' principle) is obtained. *J. B. Diaz* (College Park, Md.).

Nakamori, Kanzi. On a nonlinear boundary problem for the equation $\Delta u+cu=f(x, y)$. Mem. Fac. Sci. Kyūsyū Univ. A. **6**, 1–7 (1951). (Esperanto)

In a previous paper [same Mem. **5**, 99–106 (1950); these Rev. **13**, 650] the author proved a uniqueness and existence theorem for solutions $u(x, y)$ of the potential equation in a domain T, which have a logarithmic singularity in a point of the domain, and satisfy a boundary condition of the form $\partial u/\partial n=\varphi(x, y, u)$. Here φ as a function of u is assumed to increase monotonically from a non-positive value to infinity. The analogous theorem is proved by the same methods in the present paper for solutions u of the more general equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+c(x, y)u=f(x, y),$$

where c is non-positive and continuous in T. *F. John.*

Herglotz, Gustav. Die Greensche Funktion der Wellengleichung für eine keilförmige Begrenzung. Math. Ann. **124**, 219–234 (1952).

In 1896, A. Sommerfeld [Math. Ann. **47**, 317–374 (1896), pp. 317–341] studied the first and second boundary value problems for the wave equation $\Delta u+k^2u=0$ in connection with a wedge. The goal of the present paper is to show that for a wedge of angle π/m (m integer) solutions may be found by appeal to the theory of Green's functions through an image process. By converting the Green's function into an integral, one can treat the case of m non-integer. As special cases, the diffraction of a plane wave by a wedge and the Green's function for a half-plane and a pair of parallel plates are discussed. Some remarks are made about boundary value problems for the Laplace equation in connection with spherical lenses. *A. Heins* (Pittsburgh, Pa.).

Sambasiva Rao, P. On a series of eigenfunctions. J. Indian Math. Soc. (N.S.) **15** (1951), 140–151 (1952).

Consider the eigenvalue problem $\sum_{i=1}^{k}\partial^2u/\partial x_i^2+\lambda u=0$ for a general domain and some familiar boundary values. With the resulting eigenvalues and eigenfunctions the author sets up the series

$$(1)\qquad \sum_{m=1}^{\infty}\frac{\omega_m(P)\omega_m(Q)}{\lambda_m^{s+k/4-1/2}}J_{k/2-1}(R\sqrt{\lambda_m})$$

and he studies its analytico-meromorphic properties as a function of (P, Q, s) independent of R. The author's results are interesting, although of a pattern which is familiar by now, and he compares them with those known for the more particular series

$$(2)\qquad \sum\cdots\sum \exp\,(i(n_1h_1+\cdots+n_kh_k))J_{\beta+k/2}(2\pi\xi\sqrt{\lambda_n})\lambda_n^{-s}$$

where $\lambda_n=n_1^2+\cdots+n_k^2$, and ξ is variable.

Reviewer's note: In series (2) the Bessel functions arise "naturally" by radialization over the entire Euclidean space over which multiperiodic functions are defined; but in the case of the series (1), for a general domain, they are less cogently motivated and the introduction of a convergence factor which varies with the domain might be indicated. *S. Bochner* (Princeton, N. J.).

van den Dungen, Frans H. Sur un variant intégral associé à l'équation des ondes. C. R. Acad. Sci. Paris **235**, 532–533 (1952).

van den Dungen, Frans H. Variants intégraux associés aux équations hyperboliques linéaires. C. R. Acad. Sci. Paris **235**, 1106–1107 (1952).

Diaz, J. B., and Martin, M. H. Riemann's method and the problem of Cauchy. II. The wave equation in n dimensions. Proc. Amer. Math. Soc. **3**, 476–483 (1952).

In the present paper the authors extend a method given in a recent paper by Martin [Bull. Amer. Math. Soc. **57**, 238–249 (1951); these Rev. **13**, 244] modifying Riemann's method for the solution of the problem of Cauchy in the case of the wave equation $u_{xx}+u_{yy}-u_{tt}=0$ to the wave equation

$$(*)\qquad u_{x_1x_1}+\cdots+u_{x_nx_n}-u_{tt}=0$$

in n dimensions ($n\geqq 2$). Starting with the polar coordinates $\phi, \theta_1, \cdots, \theta_{n-2}, r$ in n-dimensional space the authors introduce characteristic coordinates $\alpha=t+r$, $\beta=t-r$; $\alpha=$const., $\beta=$const. are characteristic half-cones for the wave equation

$$\mathfrak{L}(u)=\tfrac{1}{4}f(u_{tt}-\Delta_2(u))=0.$$

With the operator $\mathfrak{L}(u)$ is associated the operator

$$M(v)=v_{\alpha\beta}+\frac{\frac{1}{2}(n-1)}{\alpha-\beta}(v_\alpha-v_\beta).$$

The authors get a generalization of Green's identity and a surface integral I_n taken around a closed n-dimensional surface S_n, vanishing whenever u, v are regular solutions of $\mathfrak{L}(u)=0$ and its associate equation $M(v)=0$, respectively. For v can be taken a special solution of the associate equation $M(v)=0$ termed the resolvent and playing the role of "Riemann's function". By application of the lemma regarding I_n and by the resolvent v, Cauchy's problem formulated in the case of the equation (*) can be solved. *M. Pinl.*

Weinstein, Alexandre. Sur le problème de Cauchy pour l'équation de Poisson et l'équation des ondes. C. R. Acad. Sci. Paris **234**, 2584–2585 (1952).

Let x denote the system of m variables $x_1, x_2, \cdots, x_m$, so that $u(x, t)$ denotes $u(x_1, x_2, \cdots, x_m, t)$. Let k be any real constant. Then the author states without proof results concerning the determination of a solution $u^k(x, t)$ of the equation

$$\Delta u^k=u_{tt}^k+kt^{-1}u_t^k$$

which satisfies the Cauchy condition $u^k=f(x)$, $u_t^k=0$ when $t=0$; here Δ denotes Laplace's operator with respect to the variables x. The solution depends on k in the manner indicated by the recurrence formula $u_t^k=tu_t^{k+2}$ if the initial conditions are such that $(k+2)u^{k+2}=\Delta u^k$, $u_t^{k+2}=0$ when $t=0$.

There are three cases. If $k=m-1$, the solution is the mean value of f on a sphere of radius t and centre x in a space of m dimensions, the generalisation of Poisson's solution of the equation of spherical waves. If $k>m-1$,

$$u^k(x, t)=\frac{\omega_{k+1-m}}{\omega_{k+1}}\int f(x_1+\alpha_1 t, \cdots, x_m+\alpha_m t)\times(1-\alpha_1^2-\cdots-\alpha_m^2)^{(k-1-m)/2}d\alpha_1\cdots d\alpha_m,$$

where integration is over $\alpha_1^2+\cdots+\alpha_m^2\leqq 1$ and ω_s is the total solid angle in space of s dimensions. Lastly, if $k<m-1$ and $k\neq -1, -3, -5, \cdots$, a repeated application of the recurrence formula leads to the solution in terms of a finite series and a simple integral involving u^{k+2n}. In the exceptional cases, $k=-1, -3, \cdots$, the problem of Cauchy has a

solution only if f is a polyharmonic function of order $(1-k)/2$ at most, and the solution is then a polynomial.

E. T. Copson (St. Andrews).

Weinstein, Alexander. On Tricomi's equation and generalized axially symmetric potential theory. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 348–358 (1951).

Tricomi's partial differential equation $\sigma Z_{\theta\theta}+Z_{\sigma\sigma}=0$, elliptic in the upper half-plane $\sigma>0$ and hyperbolic in the lower half-plane $\sigma<0$, is the simplest example of an equation of mixed type. In earlier papers [Trans. Amer. Math. Soc. **63**, 342–354 (1948); Bull. Amer. Math. Soc. **55**, 520 (1949); Naval Ordnance Lab. Rep. **NOLR-1132**, 73–82 (1950); these Rev. **9**, 584; **12**, 875] the author pointed out that for $\sigma>0$ Tricomi's equation can be transformed into an equation of generalized axially symmetric potential theory. The aim of this paper is to use these results in the investigation of the fundamental solutions and other basic solutions of Tricomi's equation and of related equations. Putting

$$x=\theta,\quad y=\tfrac{2}{3}\sigma^{3/2}\quad (y\geqq 0)\qquad\text{and}\qquad t=\theta,\quad r=\tfrac{2}{3}(-\sigma)^{3/2},$$

respectively, Tricomi's equation becomes:

$$Z_{xx}+Z_{yy}+\frac{1}{3y}Z_y=0\quad\text{and}\quad Z_{tt}-Z_{rr}-\frac{1}{3r}Z_r=0,$$

respectively. In place of these equations the author discusses the generalized types:

$$Z_{xx}+Z_{yy}+\frac{k}{y}Z_y=0\quad\text{and}\quad Z_{tt}-Z_{rr}-\frac{k}{r}Z_r=0$$

(k = real constant).

The fundamental solution of Tricomi's equation can be represented in closed form by the integral

$$Z=S_{-2/3}\int_0^\pi[(\theta+\tfrac{2}{3}(-\sigma)^{3/2}\cos\alpha)^2+\tfrac{4}{9}\beta^3]^{-1/6}\sin^{-2/3}\alpha d\alpha,$$

$$S_{p-1}^{-1}=\int_0^\pi \sin^{p-1}\alpha d\alpha,\quad b=\tfrac{2}{3}\beta^{3/2}$$

($x=0$, $y=b$ singular point of the solution). *M. Pinl.*

Germain, Paul, et Bader, Roger. Sur quelques problèmes aux limites, singuliers, pour une équation hyperbolique. C. R. Acad. Sci. Paris **231**, 268–270 (1950).

In this paper a method is developed which recovers immediately some results of Tricomi and Frankl regarding a Cauchy problem for the differential equation

$$k(\sigma)\psi_{\theta\theta}+\psi_{\sigma\sigma}=0\quad (k(\sigma)\neq 0,\text{ if }\sigma\gtreqless 0\text{ and }k(\sigma)=O(\sigma)\text{ if }\sigma\to 0).$$

The Riemann function used by the authors leads to the discussion of the analytical extension of this hypergeometric function. *M. Pinl* (Dacca).

Germain, Paul. Nouvelles solutions de l'équation de Tricomi. C. R. Acad. Sci. Paris **231**, 1116–1118 (1950).

The partial differential equation of Tricomi is invariant under translations parallel to a coordinate axis (in suitable independent variables). Hence the author gets a group G depending on three parameters making it possible to get new solutions from a given solution. Some of these solutions were given earlier by Darboux, others are new. Two subgroups G_1 and G_2 of G are also discussed. *M. Pinl.*

Germain, Paul, et Bader, Roger. Sur le problème de Tricomi. C. R. Acad. Sci. Paris **232**, 463–465 (1951).

The authors prove the existence of a solution ψ of the partial differential equation

$$\sigma\psi_{\theta\theta}+\psi_{\sigma\sigma}=0$$

in the case discussed by Tricomi [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **14**, 133–247 (1923)]. The problem under consideration is to find a solution ψ in a domain $D(\Gamma, A, B)$ limited by a simple arc Γ of the half-plane $\sigma\geqq 0$ and two characteristics AC and BC of $\sigma>0$. The points A and B are points of Γ on the axis $\sigma=0$. The function ψ is prescribed on Γ ($\psi=g$) and on one of the characteristic arcs ($\psi=f$); f and g are continuous functions satisfying the so-called condition of Hölder. The authors obtain the inequality

$$\min(m,0)\leqq\psi\leqq\max(M,0)$$

for every solution ψ in $D(\Gamma, A, B)$, if $\psi=g$ on Γ and $\psi=0$ on one of the characteristic limits of D ($m\leqq g\leqq M$). This result corresponds to the uniqueness theorem of Tricomi. The problem of Tricomi resolved in $D(\Gamma, A, B)$ can be resolved in a domain $D_1(\Gamma_1, A, B)\supset D(\Gamma, A, B)$ if Γ and $\sigma=0$ have only a finite number of common points with vertical tangents in these points. *M. Pinl* (Dacca).

Germain, P. Recherches sur une équation du type mixte. Introduction à l'étude mathématique des écoulements transsoniques. Recherche Aéronautique no. 22, 7–20 (1951).

The differential equation of the velocity potential $\varphi(x, y)$ of the steady two-dimensional irrotational isentropic flow

$$(*)\quad (\varphi_x^2-c^2)\varphi_{xx}^2+2\varphi_x\varphi_y\varphi_{xy}+(\varphi_y^2-c^2)\varphi_{yy}=0,\quad \varphi_x=u,\quad \varphi_y=v$$

is a hyperbolic differential equation if $u^2+v^2-c^2>0$ and an elliptic differential equation if $u^2+v^2-c^2<0$. By the methods of S. A. Chaplygin the differential equation (*) can be replaced by the system

$$\frac{\partial\varphi}{\partial\theta}=\frac{2\tau}{(1-\tau)^\beta}\frac{\partial\psi}{\partial\tau},\quad \frac{\partial\psi}{\partial\theta}=\frac{-2\tau(1-\tau)^{\beta+1}}{1-(2\beta+1)\tau}\cdot\frac{\partial\varphi}{\partial\tau};$$

(ψ = stream function; $\tau q_m^2=q$; θ, q = polar coordinates in the hodograph-plan; q_m = maximal speed corresponding to the case of vanishing density; $\beta=(\gamma-1)^{-1}$; γ = adiabatic exponent). The author gets a transformation of Chaplygin's system into the simple form:

$$\frac{\partial\varphi}{\partial\theta}=-\frac{\partial\psi}{\partial\sigma};\quad \frac{\partial\psi}{\partial\theta}=\frac{1}{k(\sigma)}\frac{\partial\varphi}{\partial\sigma}.$$

Hence:

$$k(\sigma)\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial\sigma^2}=0.$$

The mathematical problem of getting suitable integrals of the differential equations of the velocity potential and stream function of steady two-dimensional irrotational isentropic flow, therefore, is intimately related with the problem of Tricomi regarding the simplest example of an equation of mixed type:

$$\sigma\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial\sigma^2}=0.$$

In the second part of this paper the author deals with the problem of Tricomi, a fundamental problem in the theory of partial differential equations of mixed type. He reports on the solution of the problem of Cauchy in the case of prescribed values along a parabolic line and the solution of

the problem of Dirichlet in a domain of the elliptic half-plane whose border contains an interval of the parabolic line. In a third part of the paper the author gives some interpretations of transonic escapes using results of Tricomi and Frankl [F. Frankl, Izvestiya Akad. Nauk SSSR. Ser. Mat. **9**, 121–143 (1945); Doklady Akad. Nauk SSSR (N.S.) **56**, 683–686; **57**, 661–664 (1947); these Rev. **7**, 496; **9**, 390, 478]. The treatment especially contains the case of subsonic and sonic jets and studies the profiles of critical escapes including flows in tubes and escapes with shock waves. Finally, singular solutions of Tricomi's equation are considered in connection with several papers of S. Bergman [cf., e.g., Amer. J. Math. **70**, 856–891 (1948); these Rev. **10**, 752].
M. Pinl (Dacca).

Chang, Chieh-Chien, and Werner, Jack. A solution of the telegraph equation with application to two dimensional supersonic shear flow. J. Math. Physics **31**, 91–101 (1952).

The telegraph equation with which the authors are concerned may be written in the general form:

$$(*) \qquad au_{tt}+2bu_t+cu=u_{xx}, \quad a>0$$

a, b, and c being constant coefficients. Since this equation is concerned with the derivative of u, it is found convenient to differentiate (*) with respect to t and to solve directly for $u_t(x,t)$ instead of for u. Differentiating and introducing the change of variable $u_t=e^{(-b/a)t}$, the equation

$$A^2(u_t')_{xx}-(u_t')_{tt}+B^2u_t'=0$$

is obtained, which can be subjected to Riemann's integration method. But Riemann's formula depending in this case on $u'_{tt}(\xi,0)$ does not yet represent the final solution, since $u'_{tt}(\xi,0)$ is not actually known explicitly. Denoting $u'_{tt}(\xi,0)$ by $g(\xi)$, Volterra's integral equation of the first kind is satisfied by $g(\xi)$. The authors solve this integral equation with the aid of the Laplace transform $\mathfrak{L}[g_1(x')]$ and its inversion $g_1(\xi')$. The evaluation of the integral defining $g_1(\xi')$ is difficult. Finally the authors get an explicit expression (expansions depending on Bessel functions) and the complete determination of $u_t'(x,t)$ in the region $x-At\leqq 1$. There follows the determination of $u(x,t)$ on the boundary $t=0$, $0\leqq x\leqq 1$ with the aid of Laplace transforms and the convolution theorem and the application to the case of supersonic shear flow:

$$a=1, \quad b=-\lambda, \quad c=0.$$

These specific values of the coefficients lead to an equation which is a special case of the linearized equations governing the perturbation pressure in a two-dimensional supersonic shear flow. In evaluating the extent to which this special case may be typical of the general phenomenon the authors study the various shapes of velocity profiles which are possible under these assumptions and give some graphical representations and numerical calculations. A figure shows the pressure distribution over a wedge airfoil with a horizontal lower surface in supersonic shear flows of both positive and negative Mach number gradients.
M. Pinl (Dacca).

Conti, Roberto. Determinazione in grande delle soluzioni di un'equazione di tipo misto della dinamica dei gas in funzione dei valori assunti sulla linea parabolica. Ann. Mat. Pura Appl. (4) **32**, 235–248 (1951).

The equation considered is

$$[\bar\omega^2\omega_m^2-\bar\omega^2u^2-\omega_m^2v^2]\psi_{uu}-2[\bar\omega^2-\omega_m^2]uv\psi_{uv} \\ +[\bar\omega^2\omega_m^2-\bar\omega^2v^2-\omega_m^2u^2]\psi_{vv}-2(\bar\omega^2-\omega_m^2)(u\psi_u+v\psi_v)=0,$$

where $\bar\omega$ and ω_m are constants satisfying $\bar\omega^2<\omega_m^2$. The equation is elliptic for $u^2+v^2<\bar\omega^2$ and hyperbolic in the ring $\bar\omega^2<u^2+v^2<\omega_m^2$. It is proved that for prescribed values of ψ on the parabolic line $u^2+v^2=\bar\omega^2$, which satisfy a Lipschitz condition of order α $(0<\alpha<1)$, there exists and is uniquely determined a corresponding solution in the domain $0\leqq u^2+v^2<\omega_m^2$. The solution in the elliptic part is constructed by means of the Fourier series expansion. For the hyperbolic domain use is made of previous results of the author on the Cauchy problem with data on the parabolic line [same Ann. (4) **31**, 303–326 (1950); these Rev. **13**, 243].
F. John (New York, N. Y.).

Owens, O. G. Homogeneous Dirichlet problem for inhomogeneous ultrahyperbolic equation. Amer. J. Math. **74**, 307–316 (1952).

The equation

$$(1) \qquad u_{x_1x_1}+u_{x_2x_2}-u_{y_1y_1}-u_{y_2y_2}=f(x_1,x_2,y_1,y_2)$$

is considered in a 4-dimensional domain G, which is the direct product of a domain $G(X)$ in the x_1x_2-plane and of a domain $G(Y)$ in the y_1y_2-plane. Let λ_n and μ_m denote respectively the eigenvalues of $G(X)$ and $G(Y)$ with respect to the two-dimensional Laplace equation. It is shown that a solution u of (1) is determined uniquely by its values on the boundary of G, provided no $\lambda_n-\mu_m$ vanishes. Moreover, if $|\lambda_n-\mu_m|$ has a positive lower bound, then a solution of (1) vanishing on the boundary of G exists, provided f has sufficiently many derivatives and vanishes of sufficiently high order on the boundary of G. An example is given of a domain $G=G(X)\times G(Y)$, for which $|\lambda_n-\mu_m|$ has a positive lower bound. *F. John* (New York, N. Y.).

Halilov, Z. I. Cauchy's problem for an infinite system of partial differential equations. Doklady Akad. Nauk SSSR (N.S.) **84**, 229–232 (1952). (Russian)

The system of equations

$$(*) \quad \frac{\partial u_i}{\partial t}=\sum_{j=1}^{\infty}\sum_{(k_s)}A_{ij}^{(k_1,\cdots,k_n)}(t)\frac{\partial^{k_1+\cdots+k_n}u_j}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}} \\ +f_i(t,x_1,\cdots,x_n),\ i=1,2,\cdots,$$

where the k_s range over all non-negative integers less than a fixed integer M, is considered in the domain $0\leqq t\leqq T$, $-\infty<x_s<\infty$, $x=1,2,\cdots,n$, subject to the initial conditions $u_i(x_1,\cdots,x_n)=\varphi_i(x_1,\cdots,x_n)$, for $t=t_0$. The functions $A_{ij}^{(k_1,\cdots,k_n)}$, f_i, φ_i, u_i are complex-valued functions of the corresponding real variables. The result states that the Cauchy problem for (*) is correctly set when the $A_{ij}^{(k_1,\cdots,k_n)}$ satisfy a certain regularity condition involving bounds for the solution of an associated infinite system of homogeneous ordinary differential equations and when the f_i and φ_i satisfy certain smoothness and convergence criteria. Bounds for the derivatives of the u_i are given. *M. H. Protter*.

Integral Equations

✱Gantmaher, F. R., and Krein, M. G. Oscillyacionnye matricy i yadra i malye kolebaniya mehaničeskih sistem. [Oscillation matrices and kernels and small oscillations of mechanical systems.] 2d ed. Gosudarstv. Isdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 359 pp. 16.50 rubles.

The authors develop in this significant book an extensive theory relating largely to sets of characteristic functions

$\phi_i(x)$ with (inter alia) the Čebyšev property that a linear combination of $\phi_m(x), \cdots, \phi_n(x)$ $(m \leqq n)$ has between m and n zeros in the relevant x-interval. The theory has contact with previous work of Perron on matrices with positive elements [Math. Ann. 64, 1–76 (1907)], of Sturm on the zeros of eigen-functions and of polynomials, and of O. D. Kellogg on certain integral equations [Amer. J. Math. 38, 1–5 (1916); 40, 145–154 (1918)], but is mainly original. There appear to be substantial changes as compared with the first edition [these Rev. 3, 242]. Some of the material has been published elsewhere. After a detailed introduction the authors give, in Chapter I, a review (66 pages) of matrix theory, particularly spectral theory, adapted to the unconventional purpose of the book. Chapter II, "oscillation matrices", opens, by way of introduction, with the special case of normal Jacobian matrices, i.e., matrices $\|a_{ik}\|$ such that $a_{ik}=0$ if $|i-k|>1$, and $a_{ik}<0$ if $|i-k|=1$. Such matrices have real simple characteristic numbers, and the associated polynomials and vectors have many interesting properties regarding zeros. For the latter purpose a vector $(u_1, \cdots, u_n)$ is made into a function by joining the points $(1, u_1), \cdots, (n, u_n)$ on a Cartesian graph. Such properties are then also proved independently for "oscillation matrices", such that all the minors of all orders are non-negative and such that at least one iterate of the matrix itself is positive. The chapter concludes with a section on the variation of the characteristic numbers with the elements of an oscillation matrix. It is emphasized that the symmetry of the matrix is not essential for the basic oscillatory properties to hold. Chapter III, on small oscillations with n degrees of freedom, deals first with "Sturmian systems", characterised dynamically by the forms

$$T=\sum_{i=1}^{n} c_i \dot{q}_i^2, \quad V=\sum_{i=1}^{n} a_i q_i^2 - 2\sum_{i=1}^{n-1} b_i q_i q_{i+1}$$

$(b_i>0)$, [cf. M. Bôcher, Bull. Amer. Math. Soc. 18, 1–18 (1911), on the unpublished papers of Sturm]. Also treated are Čebyšev systems (see above) in general, and the oscillations of linear continua (e.g., bars, strings) with the statical property that under the influence of n forces the displacement cannot change sign more than $n-1$ times. Chapter IV deals with linear continua with an infinite number of degrees of freedom and Stieltjes mass-distribution. The authors consider the integral-equation approach, based on D'Alembert's principle and the influence-function, to be more natural and general than differential equation methods. They define an "oscillation kernel" $K(x, s)$ $(a \leqq x, s \leqq b)$ by the requirements

$$K(x, s)>0, \quad K\begin{pmatrix} x_1, \cdots, x_n \\ s_1, \cdots, s_n \end{pmatrix} \geqq 0, \quad K\begin{pmatrix} x_1, \cdots, x_n \\ x_1, \cdots, x_n \end{pmatrix}>0,$$

$(a<x_1<\cdots<x_n<b)$, $(a<s_1<\cdots<s_n<b)$ $(n=1, 2, \cdots)$. For such kernels the eigen-functions and eigen-values have the familiar oscillatory properties. Other topics in this chapter are oscillation theorems for forced oscillations, and binary kernels of the form

$$K(x, s)=\psi(x)\chi(s) \quad (x \leqq s), \quad =\psi(s)\chi(x) \quad (x \geqq s),$$

including the case of Sturm-Liouville problems. Chapter VI generalises some of the results for oscillation matrices to "sign-definite" matrices [see review of 1st edition]. There are two appendices, the first giving an approximate method for calculating the characteristic numbers and vectors of an oscillation matrix, while the second explains the solution of the inverse problem of the vibrating string with prescribed frequencies by means of continued fractions. The book is characterised throughout by a clear style, by a wealth of results, and by a close union between the mathematical and the dynamical aspects of the investigation.

F. V. Atkinson (Ibadan).

Harazov, D. F. Some properties of the characteristic functions and resolvent of integral equations with kernels rational with respect to a parameter. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 205–210 (1947). (Russian)

The author considers the integral equation

$$u(x)-\int_a^b G(x, y; \lambda)u(y)dy=0,$$

where either (1) $G=\sum_0^m G_n(x, y)\lambda^n$ ($G_n \varepsilon L_2(S)$, where S is the square $a \leqq x, y \leqq b$) or

$$(2) \qquad G=K(x, y)+\lambda \sum_{i=1}^{p} \sum_{k=1}^{m_i} (\lambda-\lambda_i)^{-k} G_k^{(i)}(x, y).$$

In the case (1) the conditions (G) are that the real G_n form an orthogonal set [$\int G_i(x, s)G_j(s, y)ds=0$ for $i \neq j$ almost everywhere on S] and that $\int\int G_0(x, y)\phi(x)\phi(y)dxdx<1$ for $\|\phi\|=1$. In the case (2) the conditions (G*) are as follows: K, $G_k^{(i)}$ are real, belong to $L_2(S)$, and are mutually orthogonal; the λ_j, $\neq 0$, are real; $\int\int K(x, y)\phi(x)\phi(y)dxdy<1$ when $\|\phi\|=1$. A number of theorems are established under the above hypotheses. Some of the results under hypothesis (G*) are as follows. In order that λ_0 be a c.v. (characteristic value) of G it is necessary and sufficient that for a pair (t, n) $(1 \leqq t \leqq p, 1 \leqq n \leqq m_t)$ $(\lambda_0-\lambda_t)^{-n}\lambda_0$ be a c.v. of $G_n^{(t)}$; every c.f. (characteristic function) of $G_k^{(i)}$, belonging to the c.v. $(\lambda_0-\lambda_i)^{-k}\lambda_0$, is a c.f. of G corresponding to the c.v. λ_0, and every c.f. ϕ of G belonging to the c.v. λ_0 is of the form $\phi=\lambda_0\sum\sum(\lambda_0-\lambda_i)^{-k}\psi_k^{(i)}$, where the $(\lambda_0-\lambda_i)^{-k}\lambda_0$ and the $\psi_k^{(i)}$ are c.v.'s and c.f.'s of $G_k^{(i)}$; if a function $G_1^{(i)}$ is symmetric, there exists a real c.v. of G. Also is considered the special case of hypothesis (G*) when $m_1=\cdots=m_p=1$.

W. J. Trjitzinsky (Urbana, Ill.).

Harazov, D. F. On linear integral equations whose kernel is a polynomial of the second degree in a parameter. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 275–281 (1947). (Russian)

The author studies the equation

$$(1) \qquad u(x)-\int_a^b G(x, y; \lambda)u(y)dy=0,$$

where (2) $G=G_0+\lambda G_1+\lambda^2 G_2$, with G_j real and in $L_2(S)$ (S being the square $a \leqq x, y \leqq b$). Hypothesis (K) is: (a) the G_j are orthogonal on S (in the sense of the preceding review), $\int\int G_0(x, y)\phi(x)\phi(y)dxdy<1$ for $\|\phi\|=1$. If $G \varepsilon$ (K) and the G_j are symmetric, G will have real c.v.'s (characteristic values) if and only if the kernel $G_2(x, y)$ is semi-definite positive; under these conditions the resolvent of G will have only simple poles. It is said that G satisfies hypothesis (K*) if the G_j are symmetric, G_1 and G_2 are orthogonal on S to G_0, $\int\int G_0(x, y)\phi(x)\phi(y)dxdy<1$ for $\|\phi\|=1$ and G_2 is semi-definite positive. Let

$$[\phi', \phi''; \lambda', \lambda''] = \int \phi'\phi''dx+\lambda'\lambda''\int\int G_2(x, y)\phi'(x)\phi''(y)dxdy;$$

the c.v.'s $\lambda_1, \lambda_2, \cdots$ of G, $G \varepsilon$ (K*), are real with no finite limit points; the c.f.'s (characteristic functions) of G can be chosen orthogonal in the sense that $[\phi_m, \phi_n; \lambda_m, \lambda_n]=0$ $(m \neq n)$, $=1$ $(m=n)$. If $G \varepsilon$ (K*) and G_2 is continuous, then

$\sum\lambda_n^{-2}\phi_n^2(x)$ is boundedly convergent a.e. (almost everywhere) to a function in $L_1(a, b)$; under these conditions $\sum\lambda_n^{-2}\phi_n(x)\phi_n(y)$ is absolutely convergent a.e. to a function in $L_2(S)$, the series $\sum\lambda_n^{-2}[\int H(x, y)\phi_n(y)\phi_n(y)dy]^2$ $(H \varepsilon L_2(S))$ converges a.e. to a function in $L_2(a, b)$ and the series $\sum\lambda_n^{-2}\int\int H(x, y)\phi_n(x)\phi_n(y)dxdy$ converges absolutely.
W. J. Trjitzinsky (Urbana, Ill.).

Harazov, D. F. On the theory of linear integral equations with kernels polynomial in a parameter. Soobščeniya Akad. Nauk Gruzin. SSR. 9, 91–98 (1948). (Russian)

The author studies the equation

$$(1) \qquad u(x) - \int_a^b G(x, y; \lambda)u(y)dy = f(x),$$

where (2) $G = \sum_0^m \lambda^n G_n(x, y)$, the $G_n(x, y)$ are real and in $L_2(S)$ $[S = (a \leqq x, y \leqq b)]$, f is real and in $L_2(a, b)$. Hypothesis (G) consists of the conditions: (a) the G_n are orthogonal on S in the sense of the author's paper reviewed second above, referred to as (H); (b) $\int\int G_0(x, y)\phi(x)\phi(y)dxdy < 1$ when $\|\phi\| = 1$. The present work constitutes a continuation of (H). If $G \varepsilon$ (G) and λ is not a c.v. (characteristic value), then the resolvent of G is meromorphic in λ and is of the form $R(x, y; \lambda) = \sum_0^m \lambda^n R_n(x, y; \lambda^n)$, where $R_n(x, y; \mu)$ is the resolvent of G_n. If $K(x, y)$ is symmetric and is in $L_2(S)$, its resolvent Γ is represented by a series [in terms of c.v.'s λ_ν and orthonormalized c.f.'s (characteristic functions) ϕ_ν], which converges a.e. (almost everywhere) on S to Γ and also converges in the mean square. If $G \varepsilon$ (G), λ is not a c.v. of G and the G_n are symmetric, then the resolvent R of G is representable a.e. by a series in terms of G and of the c.v.'s $\lambda_j^{(n)}$ and the orthonormalized c.f.'s $u_j^{(n)}(x)$ of the G_n; this series also converges in the mean square; in this case every c.v. of G is a simple pole of R. If $G \varepsilon$ (G), the G_n are symmetric and λ is not a c.v. of G, then the solution of (1) is given by $u(x) = f(x) + \sum_{k=0}^m \lambda^k u_k(x)$, where $u_k(x) = \sum_i(\lambda_i^{(k)} - \lambda^k)^{-1} f_i^{(k)} u_i^{(k)}(x)$, $f_i^{(k)} = \int_a^b f u_i^{(k)} dx$, convergence to $u(x)$ taking place a.e.; convergence is also in the mean square. There is also an analogue to the third theorem of Fredholm. *W. J. Trjitzinsky* (Urbana, Ill.).

Harazov, D. F. On linear integral equations with generalized Schmidt kernels. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 16, 143–167 (1948). (Russian. Georgian summary)

The author studies the system

$$(1) \qquad u(x) = \int_a^b [K_0(x, y) + \lambda K_1(x, y)]v(y)dy,$$

$$v(x) = \int_a^b [K_0(y, x) + \lambda K_1(y, x)]u(y)dy,$$

where K_0, K_1 are real in $L_2(S)$, $S = E[x, y \mid a \leqq x, y \leqq b]$. For $K_1(x, y) \equiv 0$, (1) becomes (1_0), which is a system considered by E. Schmidt. One obtains

$$(2) \qquad u(x) = \int_a^b [\bar{G}_0(x, t) + \lambda\bar{G}_1 + \lambda^2\bar{G}_2]u(t)dt,$$

$$(3) \qquad v(x) = \int_a^b [\underline{G}_0(x, t) + \lambda\underline{G}_1 + \lambda^2\underline{G}_2]v(t)dt,$$

where the $\bar{G}_i$, $\underline{G}_i$ are symmetric and are in $L_2(S)$. The c.v.'s (characteristic values) of (2), (3) are identical. If λ^* is a c.v. of (2) and $\varphi^*(x)$ is a corresponding c.f. (characteristic function), then λ^* is a c.v. of (1) and $\varphi^*(x)$ and

$$\psi^*(x) = \int [K_0(t_1, x) + \lambda^* K_1(t_1, x)]\varphi^*(t_1)dt_1$$

satisfy (1) for $\lambda = \lambda^*$. If (4) $\int[\int K_0(x, y)\varphi(x)dx]^2 dy < 1$ when $\|\varphi\| = 1$, then (1) has at least one c.v., all c.v.'s are real and have no finite limit points. If $K_0(x, y)$, $K_1(x, y)$ are orthogonal on S in the sense of Goursat and (4) holds, then (1) is equivalent to (1_0). The author proceeds to determine the c.v.'s and c.f.'s following the lines of Goursat [Cours d'analyse, t. III, 3ième ed., Gauthier-Villars, Paris, 1921] and Courant [Math. Ann. 89, 161–178 (1923)], employing sequences of degenerate kernels. In the course of such developments certain sequences $(\lambda_i^{(m)})$ of c.v.'s are obtained; conditions on these sequences are then found, under which an effective construction of the total spectrum of c.v.'s of (2) is possible. *W. J. Trjitzinsky* (Urbana, Ill.).

Elliott, Joanne. On a class of integral equations. Proc. Amer. Math. Soc. 3, 566–572 (1952).

Given a measurable function $w(t) \geqq 0$, write $H[w]$ for the Hilbert space of measurable functions $f(t)$ defined in $(-1, 1)$ such that

$$\int_{-1}^1 |f(t)|^2 w(t)dt < \infty.$$

The author shows by using Jacobi polynomials that if $w(t) = (1-t)^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}$, $w_1(t) = 1/w(t)$, the transformation

$$Tf = \frac{1}{\pi} P \int_{-1}^1 \frac{f(t)w(t)}{t-x} dt$$

(P denoting the Cauchy principal value) maps $H[w]$ isometrically into $H[w_1]$; the inverse mapping is

$$T_1 g = -\frac{1}{\pi} P \int_{-1}^1 \frac{g(t)w_1(t)}{t-x} dt.$$

These results are used to transform the singular integral equation

$$g(x) = \frac{1}{\pi} P \int_{-a}^a \frac{f(t)}{t-x} dt + \lambda \int_{-a}^a h(x, t)f(t)dt$$

into a standard Fredholm equation, under appropriate conditions on $h(x, t)$ and $g(x)$. The equation with $a = \infty$ is treated similarly by using Hilbert transforms.
F. Smithies (Cambridge, England).

Pogorzelski, W. Sur la solution de l'équation intégrale dans le problème de Fourier. Ann. Soc. Polon. Math. 24 (1951), 56–74 (1952).

The author reduces the two-dimensional heat equation with a mixed linear boundary condition to an integral equation of the form

$$\mu(s, t) = f(s, t) + \lambda \int_0^t d\tau \int_C N(s, t; \sigma, \tau)\mu(\sigma, \tau)d\sigma,$$

where C is a simple closed curve, and the kernel is

$$N(s, t; \sigma, \tau) = \left\{\frac{\Gamma_{s\sigma} \cos \varphi_{s\sigma}}{2(t-\tau)^2} + \frac{a(s, t)}{t-\tau}\right\} \exp[-\Gamma_{s\sigma}^2/4(t-\tau)],$$

$\varphi_{s\sigma}$ being the angle between the tangents at the points s and σ of the curve C. Assuming that the direction of the tangent, as a function of distance along the arc, satisfies a Hölder condition, the author proves that one of the iterates of the kernel is bounded, and then uses the fact that the equation

is of Volterra type to show that the Neumann series is convergent and represents the solution of the equation for all $t>0$ and all complex λ.

In the second part of the paper a similar result is obtained for the corresponding n-dimensional problem. The conditions on the bounding hypersurface, though complicated, are analogous to the Hölder condition mentioned above. The necessary estimations to prove the boundedness of an iterated kernel are obtained by using some ingenious devices.

F. Smithies (Cambridge, England).

Pogorzelski, W. Sur l'équation intégro-différentielle non linéaire à singularité polaire. Ann. Soc. Polon. Math. 24 (1951), 75–87 (1952).

The author considers an equation of the form

$$P\int_0^a N(x,y)F[x,y,\varphi(y),\varphi'(y),\cdots,\varphi^{(n)}(y)]dy=f(x),$$

where $N(x,y)=M_1(x,y)\cot[\pi(x-y)/a]+M_2(x,y)$, M_1 and M_2 being holomorphic functions of x and y in a strip containing the real axis, $F(x,y,u_0,\cdots,u_n)$ being holomorphic in all $n+3$ variables, and $f(x)$ being holomorphic in x; also all functions are supposed periodic in x and in y with real period a. The notation P indicates that the integral is to be taken in the principal value sense. By transforming the equation into a regular integro-differential equation of the form

$$\Phi[x,\varphi(x),\varphi'(x),\cdots,\varphi^{(n)}(x)]$$
$$=\int_0^a \Psi[x,y,\varphi(y),\varphi'(y),\cdots,\varphi^{(n)}(y)]dy,$$

and then differentiating, the author shows that, under certain assumptions, the equation is equivalent to a system of equations of the form

$$\varphi'(x)=\psi_1(x),\quad \psi_1'(x)=\psi_2(x),\ \cdots,\ \psi'_{n-1}(x)=\psi_n(x),$$
$$\psi_n'(x)=\Omega[x,\varphi(x),\psi_1(x),\psi_2(x),\cdots,\psi_n(x)].$$

Under certain further assumptions, these can be solved by a process of successive approximation. *F. Smithies.*

Hvedelidze, B. V. Some properties of improper integrals in the sense of the Cauchy-Lebesgue principal value. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 283–290 (1947). (Russian)

Let C be a smooth, closed or open curve in the complex plane (the angle of the tangent line being of Hölder class as function of point on C); C is given by equations $x=x(s)$, $y=y(s)$ $(0\leqq s\leqq 2\pi)$, where s is length of arc; length of C is equal to 2π (the latter condition entailing no loss of generality). If complex-valued $f(t)$ is defined for t on C, let $f_*(s)=f(t(s))$ $(t(s)=x(s)+iy(s))$; $f_*(s)\in L^p[0,2\pi]$ or $f(t)\in L^p[C]$, if $\int_0^{2\pi}|f_*(s)|^p ds<+\infty$. The symbols $H[0,2\pi]$, $H[C]$, $H_\alpha[C]$ ($\alpha=$Hölder exponent) are used to designate Hölder classes. If t_0 is on C and $f\in L^1[C]$, the integral

$$(1)\qquad \phi(t)=\int_C f(t)\frac{dt}{t-t_0}$$

is considered in the sense of Cauchy-Lebesgue principal values (if it exists in this sense). The purpose of the author is to establish some of the properties of such integrals, with a view to eventual generalization of the existing theory of integral equations in the sense of principal values. Some of the results are as follows. If $f\in L^1[C]$, then $\phi(t)$ is finite a.e. (almost everywhere) on C; if $f\in L^p[C]$ $(p>1)$, then $\phi\in L^p[C]$. If $K(t,\tau)\in H[C]$ and $f\in L^p[C]$, $p>1$, then

$$\int_C Kf(\tau)\frac{d\tau}{\tau-t}\in L^p[C].$$

If $f\in L^p$, $\phi\in L^q$, $p>1$, $q=p/(p-1)$, and $K(t,\tau)\in H[C]$, then the order of integration in

$$\int_C\int_C K(t,\tau)f(t)\phi(\tau)\frac{dtd\tau}{t-t_0}$$

is immaterial a.e. on C. If $f\in L^p$, $\phi\in L^q$, then a.e. on C one has

$$\int_C\frac{f(t)dt}{t-t_0}\int_C\frac{\phi(\tau)d\tau}{\tau-t}=\pi^2 f(t_0)\phi(t_0)+\int_C\phi(\tau)d\tau\int_C\frac{f(t)dt}{(t-t_0)(\tau-t)}.$$

W. J. Trjitzinsky (Urbana, Ill.).

Hvedelidze, B. V. Singular integral equations in improper Cauchy-Lebesgue integrals. Soobščeniya Akad. Nauk Gruzin. SSR. 8, 427–434 (1947). (Russian)

Use is made of the notation and results of an earlier paper of the author [see the preceding review], referred to as (H). Let $C=\sum C_k$ be a set of simple, smooth (as in (H)) nonintersecting curves, forming the boundary of a connected domain D^+; along C the direction assigned is that leaving D^+ on the left; the origin is in D^+; D^- is the complement of D^++C. On making largely use of (H), the author establishes the essential features of the theory of integral equations in the sense of Cauchy-Lebesgue principal values, when integrations are along C, thereby generalizing the existing theory. Let

$$I\phi=\frac{1}{\pi i}\int_C\frac{\phi(\tau)d\tau}{\tau-t},\quad T\phi=\int_C T(t,\tau)\phi(\tau)d\tau,$$

$$K\phi\equiv A(t)\phi(t)+B(t)I\phi+T\phi,$$

where A, B, T are of Hölder classes, $A^2-B^2\neq 0$ on C; the adjoint of K is $K'\psi\equiv A(t)\psi+IB\psi+\int_C T(\tau,t)\psi(\tau)d\tau$. If the integral equation (1) $K\phi=f(t)\in L^p[C]$ $(p>1)$ has a solution in $L^p[C]$, then $\int f\psi dt=0$, where ψ is any solution $L^q[C]$ of $K'\psi=0$. If K_1, K_2 are operators of the type of K, there is an explicit composition formula for $K^*\equiv K_1K_2$; this leads to regularization. Thus, if the equation is (1), with $f\in L^2[C]$, this equation can be regularized so that the new equation $K^*\psi=K_1f$ is a regular Fredholm equation; one obtains theorems of Noether's type. *W. J. Trjitzinsky.*

Hvedelidze, B. V., On Riemann's problem in the theory of analytic functions and singular integral equations with kernels of Cauchy type. Soobščeniya Akad. Nauk Gruzin. SSR. 12, 69–76 (1951). (Russian)

The present work is largely based on earlier papers of the author [see the two preceding reviews]; the latter will be referred to as (H_2). In (H_2) the Riemann boundary problem for analytic functions was generalized, when the contours $C=\sum C_k$ (simple, suitably smooth, non-intersecting) are closed and bound a domain of finite connectivity. In the present developments the contours are as in (H_2), but the methods are adaptable to eventual consideration of open arcs. In (H_2) Noether-type theorems have been obtained for singular integral equations in the class $L_2(C)$. Such theorems are now obtained in $L_p(C)$ $(p>1)$. The theory of the Riemann problem $\Phi^+(t)=G(t)\Phi^-(t)+g(t)$, $g\in L_p(C)$, on

C, is developed under conditions on $G(t)$ considerably lighter than has been the case in (H); there are corresponding extensions for integral equations. *W. J. Trjitzinsky.*

Vekua, N. P. On a generalized system of singular integral equations. Soobščeniya Akad. Nauk Gruzin. SSR. **9**, 153–160 (1948). (Russian)

The author makes use of his earlier work [same Soobščeniya **8**, 577–584 (1947); these Rev. **14**, 151], referred to as (V). Let L, $\alpha(t)$, $\beta(t)$ have the same meaning as in (V). The author studies the system of integral equations

$$\sum_{k=1}^{n} A_{jk}(t_0)\mu_k[\alpha(t_0)]+\sum_{k=1}^{n} B_{jk}(t_0)\mu_k(t_0)$$
$$+\sum_{k=1}^{n}\frac{1}{\pi i}\int_L K_{jk}\, t_0, t)\mu_k(t)\frac{dt}{t-\alpha(t_0)}$$
$$+\sum_{k=1}^{n}\frac{1}{\pi i}\int_L K^*_{jk}(t_0,t)\,\mu_k(t)\frac{dt}{t-t_0}=f_j(t_0),\quad j=1,\cdots,n,$$

on L; the μ_k are unknowns to be found in (H) (Hölder class on L), the other functions are known and are in (H); $\det(A_{j,k})$, $\det(B_{j,k})$ non-zero on L. On making use of the results in (V), relating to a certain generalized Hilbert boundary value problem, the theory of the above system is developed in its essential features, leading to theorems of Noether type. *W. J. Trjitzinsky* (Urbana, Ill.).

Ahlfors, Lars V. Remarks on the Neumann-Poincaré integral equation. Pacific J. Math. **2**, 271–280 (1952).

The Neumann method for solving the integral equation associated with the boundary value problems of the first and second kinds is applied to a generalized integral equation. If the operator which occurs in the Neumann series for the solution of this problem is denoted by T, the convergence of the series depends on the energy-norm of T. The principal result is: "If B_i and B_e are complementary simply connected regions bounded by a Jordan curve C, and there exists a quasi-conformal mapping of B_i onto B_e which leaves C fixed and whose maximal eccentricity is $\leqq k$, then the norm of the transformation T is at most k." This theorem is applied to a linear integral equation which occurs in the calculation of the conformal mapping function of a star-shaped region to give the result that the associated Neumann series has a convergence ratio not exceeding the cosine of the minimum angle between the tangent and the radius vector. The case where the norm is taken as the maximum density of the mass distribution on C is also discussed.

C. Saltzer (Cleveland, Ohio).

Royden, H. L. A modification of the Neumann-Poincaré method for multiply connected regions. Pacific J. Math. **2**, 385–394 (1952).

The determination of a function analytic in a given multiply connected region whose real part is prescribed on the boundary is considered. An integral equation of the same type as the integral equations of potential theory is derived. The convergence of the solution of this equation by iteration is established by the use of the principal theorem of the above paper. The kernel of the equation for a doubly connected region is given explicitly in terms of theta functions and is applied to the problem of calculating the function which maps a doubly connected region on an annular region bounded by concentric circles. *C. Saltzer.*

Gagaev, B. M. Existence theorems for solutions of integro-differential equations. Doklady Akad. Nauk SSSR (N.S.) **85**, 469–472 (1952). (Russian)

The author formulates existence theorems for solutions of the system

$$\frac{dz_i}{dx}=P_i(x, z_1(x), \cdots, z_n(x))$$
$$+\lambda\int_0^1 G_i(x, u)Q_i(u, z_1(u), \cdots, z_n(u))du,$$

$z_i(0)=0$ $(i=1, 2, \cdots, n)$, under the usual assumptions of smallness of the parameter λ, continuity or measurability in x, and continuity in $z_1, \cdots, z_n$ of the functions P_i, Q_i. If these functions satisfy a Lipschitz condition, then the solution is unique and can be obtained by the method of successive approximations. *M. Golomb* (Lafayette, Ind.).

Functional Analysis, Ergodic Theory

Nikodým, Otton Martin. Universal real locally convex linear topological spaces. Ann. Inst. Fourier Grenoble **3** (1951), 1–21 (1952).

This partly expository paper gives a geometric proof of a corollary to the Hahn-Banach theorem, and constructs a class of locally convex spaces with the property that every locally convex space is isomorphic to a subspace of a member of this class. *E. Michael* (Chicago, Ill.).

Donoghue, William F., Jr., and Smith, Kennan T. On the symmetry and bounded closure of locally convex spaces. Trans. Amer. Math. Soc. **73**, 321–344 (1952).

Après avoir rappelé un certain nombre de résultats connus de la théorie des espaces localement convexes, les auteurs étudient un processus général de "limite inductive": étant donnés un espace vectoriel E et une famille (F_α) d'espaces localement convexes, ainsi que, pour tout α, une application linéaire continue f_α de F_α dans E, on considère sur E la topologie localement convexe la plus fine rendant toutes les f_α continues. Ils montrent que toute limite inductive d'espaces bornologiques est bornologique, et réciproquement que tout espace bornologique séparé s'obtient comme limite inductive (au sens précédent) d'espaces normés. Ils étudient ensuite les espaces E ayant les propriétés suivantes: 1) E est infratonnelé ("symmetric" dans la terminologie des auteurs), c'est-à-dire que toute partie fortement bornée du dual E' est équicontinue; 2) E', muni de la topologie forte, est bornologique. Beaucoup d'espaces fonctionnels usuels ont ces propriétés. Si un tel espace E est sous-espace fermé d'un espace localement convexe F, la topologie forte de E' est isomorphe à la topologie quotient de la topologie forte sur F' par l'orthogonal E^0 de E dans F'. Tout produit d'espaces de ce type est encore du même type (en ce qui concerne les théorèmes donnés sur les produits par les auteurs, certains sont dûs à Katětov [Acta Fac. Nat. Univ. Carol., Prague no. **181** (1948); ces Rev. **10**, 127; non cité dans la bibliographie]). Reprenant des résultats de Mackey [Bull. Amer. Math. Soc. **50**, 719–722 (1944); ces Rev. **6**, 70], les auteurs montrent que la question de savoir si un produit $\prod_{\alpha\in A} E_\alpha$ d'espaces bornologiques est bornologique dépend uniquement de la puissance de A, et est liée à un problème d'Ulam sur la mesure. Vient ensuite l'étude de l'espace $C(S)$ des fonctions numériques continues sur un espace localement compact S, muni de la topologie de la conver-

gence compacte; utilisant un résultat de Grothendieck (dont ils donnent une démonstration), les auteurs prouvent que si S est dénombrable à l'infini, E est métrisable et E' bornologique; par contre, si S est semi-compact et non compact (par exemple, l'espace des ordinaux de 1re classe), E n'est pas infratonnelé, bien que complet, et isomorphe à un sous-espace fermé d'un produit de c espaces de Banach (produit que l'on sait être bornologique, en vertu des résultats d'Ulam). Enfin, les auteurs donnent des démonstrations de deux théorèmes de Grothendieck (légèrement généralisés) sur les espaces (F); ils montrent que si E et F sont deux espaces (F) tels que E' soit bornologique, toute transformation faiblement continue de F' sur E' est un homomorphisme, et enfin que pour tout espace (LF) réflexif E, E' est bornologique. *J. Dieudonné.*

Smith, Marianne Freundlich. The Pontrjagin duality theorem in linear spaces. Ann. of Math. (2) 56, 248–253 (1952).

If G is a topological group, then $\hat{G}$ denotes the group of characters on G, with the topology of uniform convergence on compact sets. If F is a topological linear space over K, where K is the reals or the complexes, then F^{*k} (resp. F^{*b}) denotes the linear space of continuous linear functionals from F to K, with the topology of uniform convergence on compact (resp. bounded) sets. Let E be a topological linear space over K. We say that E has property $\mathfrak{P}$ if $E=\overline{E}$, and that E is reflexive if $E=(E^{*b})^{*b}$; here, and also below, the equal sign means that the natural mapping from the first space into the second is one-to-one, onto, and bicontinuous. The author proves the following results. (1) If K is the reals, then $\hat{E}$ and E^{*k} are isomorphic as topological groups. (2) If K is the reals, then E has property $\mathfrak{P}$ if and only if $E=(E^{*k})^{*k}$. (3) If E is reflexive or a Banach space, then $E=(E^{*k})^{*k}$. [Reviewer's note: Concerning (3), it can be shown that $E=(E^{*k})^{*k}$ whenever E is locally convex, tonnelé [N. Bourbaki, Ann. Inst. Fourier Grenoble 2, 5–16 (1951); these Rev. 13, 137], and such that the closed, convex hull of every compact subset of E is compact. These properties are satisfied by reflexive spaces, $\mathfrak{F}$-spaces (and hence Banach spaces), and $\mathfrak{LF}$-spaces.] *E. Michael.*

Vinokurov, V. G. On biorthogonal systems spanning a given subspace. Doklady Akad. Nauk SSSR (N.S.) 85, 685–687 (1952). (Russian)

The author announces without proof a number of theorems about the connection between the bases and biorthogonal sets in a Banach space E and those in its closed linear subspaces. The following are typical. Theorem 2. Let P and Q be complementary subspaces of E and let $x_1, x_2, \cdots$ be a basis for P which is part of a basis for E. Then there exists a basis $y_1, y_2, \cdots$ for Q such that the x_j and the y_k together form a basis for E. Theorem 3A. Let P and Q be quasi-complementary subspaces in E. Let $z_1, z_2, \cdots, F_1, F_2, \cdots$ be a biorthogonal system for E such that P is the closed linear span of the z_j with odd j and Q is the closed linear space of the z_j with even j. Let $a_1, a_2, \cdots$ be a sequence of scalars such that no a_j with odd j is zero. Let R be the closed linear span of the elements $a_1z_1+a_2z_2$, $a_3z_3+a_4z_4, \cdots$. Then R and Q are quasi-complements.
G. W. Mackey (Cambridge, Mass.).

Sirvint, G. Weak compactness in Banach spaces. Studia Math. 11, 71–94 (1950).

This paper was received by the editor in 1941, and was prepared for publication by A. Alexiewicz. It contains a detailed exposition of results announced earlier [C. R. (Doklady) Acad. Sci. URSS (N.S.) 28, 199–202 (1940); these Rev. 2, 221]. *V. L. Klee Jr.*

Nakamura, Masahiro. Complete continuities of linear operators. Proc. Japan Acad. 27, 544–547 (1951).

The author shows the equivalence of the requirements: (a) T is weakly completely continuous and (b) T^{**} maps E^{**} into E. Gantmacher [Mat. Sbornik N.S. 7(49), 301–308 (1940); these Rev. 2, 224] required a separability condition for this theorem. The author remarks that an appeal to Arzelà's theorem yields the result that a bounded subset M of a Banach space is precompact if and only if the operation $f \rightarrow f(x)$ is continuous on S^*, taken in the W^* topology, into $B(M)$ the Banach space of uniformly bounded functions on M. The Gelfand-Phillips theorem for precompactness is a variant of this.
D. G. Bourgin (Urbana, Ill.).

Edwards, R. E. On the weak convergence of bounded continuous functions. Nederl. Akad. Wetensch. Proc. Ser. A. 55 = Indagationes Math. 14, 230–236 (1952).

The representation $\int x(t)d\mu(t)$ of a linear functional on the space of real, bounded, continuous functions on T (a locally compact Hausdorff space, in this paper) can have either of the following interpretations: (a) μ is a finitely additive measure on T; (b) μ is a Radon measure on βT, the Čech compactification of T. If the support of the Radon measure is in $\beta T-T$, then the corresponding finitely additive measure is purely finitely additive [Yosida and Hewitt, Trans. Amer. Math. Soc. 72, 46–66 (1952); these Rev. 13, 543]. A functional L, corresponding to such a measure is characterized, essentially, by the condition $L(x) \leqq \lim \sup_{t\to\infty} x(t)$. This is used to obtain a sufficient condition for weak convergence of a sequence of functions. *M. Jerison.*

Krull, Wolfgang. Bemerkungen zur Theorie des Hilbertschen Raumes. Arch. Math. 3, 114–124 (1952).

The author defines a Stieltjes decomposition of a Hilbert space H as a function $t \rightarrow M(t)$ from a closed interval $I=[a, b]$ in the real line to subspaces of H, such that $M(a)=O$, $M(b)=H$, and $M(s) \subset M(t)$ whenever $s \leqq t$. A diagonal decomposition of a Hermitian operator A on H, associated with a real-valued continuous function f on I is a Stieltjes decomposition M such that $M(t)$ is invariant under A for all t and such that $(Ax, x)=f(t)$ whenever $t \in [a_0, b_0] \subset I$ and whenever x is a unit vector in $M(b_0)-M(a_0)$. The principal theorem is a characterization of the spectrum of A in terms of a diagonal decomposition; it asserts essentially that λ belongs to the spectrum of A if and only if there exists a value of t, not belonging to a constancy-interval of M, such that $f(t)=\lambda$.

A fixed Stieltjes decomposition, together with a partition of the underlying interval, induces a direct-sum decomposition and any two direct-sum decompositions obtained in this manner (by varying the partition) have a coarsest common refinement. This observation motivates the author to indicate a generalization of the concept of Stieltjes decomposition; the generalized concept is defined to be a family of direct-sum decompositions with the refinement property just mentioned. The last section of the paper is devoted to some comments about generalized Stieltjes decompositions, their algebraic analogs, and their possible domains of applicability.
P. R. Halmos (Chicago, Ill.).

Tagamlickiĭ, Ya. A. Generalization of a theorem of Minkowski. Uspehi Matem. Nauk (N.S.) 7, no. 2(48), 180–183 (1952). (Russian)

Let H be a real (complete) Hilbert space, and let K be a cone in H (i.e., $x, y \in K$ and $\alpha, \beta \geqq 0$ imply $\alpha x + \beta y \in K$) which is strongly closed. Let K^* denote the set of all $x \in H$ such that $(x, k) \geqq 0$ for all $k \in K$. Then $K^{**} = K$. This is in effect a generalization of a theorem of Minkowski [Gesammelte Abhandlungen, Bd. 2, Teubner, Berlin-Leipzig, 1911, pp. 131–229]. The proof given is brief and elementary. Since K^{**} is obviously weakly closed, this shows that a strongly closed cone is weakly closed. As another application, a theorem of Wiener is proved in slightly generalized form [see Wiener, The Fourier integral . . . , Cambridge Univ. Press, 1933, p. 100, Theorem 11]. *E. Hewitt.*

Fage, M. K. The rectification of bases in Hilbert space. Doklady Akad. Nauk SSSR (N.S.) 74, 1053–1056 (1950). (Russian)

A pair of sequences of subspaces $\{H_n\}$ and $\{G_n\}$ in a Hilbert space E is called a biorthogonal system in case 1) $H_n \perp G_m$ for $n \neq m$ and 2) there is an idempotent J_n whose range is H_n and whose adjoint J_n^* has range G_n, $n = 1, 2 \cdots$. A sequence $\{H_n\}$ is called a basis for E in case each x in E can be written as a sum: $x = \sum_{n=1}^{\infty} x_n$, $x_n \in H_n$. The mappings $J_n x = x_n$ are then idempotents, $J_n J_m = 0$ if $n \neq m$, and $\sum_{n=1}^{\infty} J_n x = x$ for all x. Thus the J_n are an idempotent resolution of the identity [Fage, same Doklady (N.S.) 73, 895–897 (1950); these Rev. 12, 186]. The spaces $G_n = J_n^* E$ together with the H_n constitute a biorthogonal system. Conversely, if $\{J_n\}$ is an idempotent resolution of the identity, $\{H_n = J_n E\}$ is a basis. The analog of the Gram-Schmidt process leads to the following: Let $\{H_n\}$ be a basis for E, $\{J_n\}$ the associated idempotents, $H^{(n)} = H_1 \oplus \cdots \oplus H_n$, and $J^{(n)}$ the projection on $H^{(n)}$. Then the operators $A_n = \sum_{k=1}^{n} J^{(k)} J_k^* J_k$ are the analogs of the Gram-Schmidt sums. The question of their convergence is answered by the following theorem: The A_n converge if and only if the basis $\{H_n\}$ arises from an orthogonal basis by a linear invertible transformation. (Note when the A_n converge, their limit is a linear invertible transformation of E onto itself.) *B. R. Gelbaum.*

Inzinger, Rudolf. Eine geometrische Realisierung des Hilbertschen Raumes in der Menge der stützbaren Bereiche einer Ebene. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 140–155 (1951).

Consider all functions $a(\phi)$ of period 2π and class L^2 which equal the $(C, 1)$ summation of their Fourier series $\sum_{-\infty}^{\infty} \alpha_\nu e^{i\nu\phi}$. The function $a(\phi)$ is considered a vector α in Hilbert space with $(\alpha, \beta) = i\pi^{-1} \int a(\phi) b(\phi) = \frac{1}{2} \sum \alpha_\nu \beta_{-\nu}$ as inner product. On the other hand the numbers ϕ, $a(\phi)$ determine in a plane π an oriented line with direction ϕ and (signed) distance $a(\phi)$ from the origin. Thus $a(\phi)$ defines the tangents of a domain A in π which is called "stützbar". The paper studies the mapping $\alpha \rightarrow A$. The totality of the $a(\phi)$ which for given values ϕ_0, h_0 satisfy $a(\phi_0) = h_0$ form a hyperplane $P(\phi_0, h_0)$ in Hilbert space. Since the planes $P(\phi, k)$ and $P(\phi_0, h)$ are parallel, the $P(\phi, h)$ form a one-parameter family of pencils of parallel hyperplanes. If the plane π is identified with the (x_1, x_2)-plane in Hilbert space, then the mapping $\alpha \rightarrow a(\phi) \rightarrow A$ may be described as follows: the relation $x(\phi) = a(\phi)$ determines for variable ϕ a family of hyperplanes $P(\phi, a(\phi))$ through the point α which forms a cone $K(\alpha)$ and A is the intersection of π with $K(\alpha)$.

H. Busemann (Los Angeles, Calif.).

Livšic, M. S. On the reduction of a linear non-Hermitian operator to "triangular" form. Doklady Akad. Nauk SSSR (N.S.) 84, 873–876 (1952). (Russian)

Let A be a bounded linear operator in a Hilbert space H, and resolve it into real and imaginary parts:

$$A = \tfrac{1}{2}(A + A^*) + i[(A - A^*)/2i] = \Re(A) + i\Im(A).$$

The operator A is said to belong to the class $(i\Omega)$ if $\Im(A)$ is completely continuous and the series of moduli of its eigenvalues is convergent.

The dimension r of the closed subspace E generated by the vectors $\{\Im(A)f : f \in H\}$ is called the non-Hermitian rank of A, and if p and q are respectively the numbers of positive and negative eigen-values of $\Im(A)$, (p, q) is called the non-Hermitian signature of A. The orthogonal complement G of E is called the foundation of A, and the closed subspace G_d of G orthogonal to all the subspaces $\{A^n E : n = 0, 1, 2, \cdots\}$ is called the supplementary component of the foundation; on G_d the operator A is Hermitian, so that the representation of A on this subspace can be handled by well-known methods.

The author now constructs what he calls a triangular model $\mathbf{A}$ of an operator with given non-Hermitian rank r and signature (p, q) in the following manner. Let $\mathfrak{H}_r$ be a complex Euclidean space of dimension r (finite or infinite); form the Hilbert space $\mathbf{H} = \mathbf{H}_I + \mathbf{H}_{II}$, where $\mathbf{H}_I$ is the Hilbert space formed from functions $f(k)$ of a positive integral variable with values in $\mathfrak{H}_r$ and $\mathbf{H}_{II}$ is the Hilbert space formed from functions $f(x)$ of a real variable x $(0 \leqq x \leqq l)$ with values in $\mathfrak{H}_r$, the inner product in $\mathbf{H}$ being defined by

$$\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f(k), g(k) \rangle + \int_0^l \langle f(x), g(x) \rangle dx.$$

Let $\alpha(k)$ and $\alpha(x)$ be non-decreasing bounded real functions with $\alpha(x-0) = \alpha(x)$, and let $\beta(k)$, $\beta(x)$ be non-negative definite Hermitian matrices of order r and finite norm for each value of k and x. The operator $\mathbf{A}$ is now defined by $\mathbf{A}f = g = (g_I, g_{II})$, where

$$g_I = g(k) = f(k)[\alpha(k) + \tfrac{1}{2} i\beta(k) J\beta(k)] + i \sum_{j=k+1}^{\infty} f(j)\beta(j) J\beta(k) + i \int_0^l f(t)\beta(t) J\beta(k) dt,$$

$$g_{II} = g(x) = \alpha(x) f(x) + i \int_x^l f(t)\beta(t) J\beta(x) dt,$$

where J is a matrix of order r of the form $[\pm\delta_{kj}]$ with p positive and q negative diagonal elements.

The author now states the following result. If

$$(1) \qquad \sum_{k=1}^{\infty} \operatorname{Tr}[\beta^2(k)] < \infty, \quad \operatorname{Tr}[\beta^2(x)] = 1 \quad (0 \leqq x \leqq l),$$

and

$$(2) \qquad u \in \mathfrak{H}_r, \quad u\beta(k) = 0 \ (k = 1, 2, \cdots), \quad u\beta(x) = 0 \ (0 \leqq x \leqq l)$$

together imply $u = 0$, then $\mathbf{A}$ belongs to the class $(i\Omega)$, and has rank r and signature (p, q); its non-real spectrum consists of the points of the form

$$\lambda_k = \alpha(k) + \tfrac{1}{2} i\gamma(k)$$

where $\gamma(k)$ runs over the non-zero eigen-values of $\beta(k) J\beta(k)$, and its real spectrum is the set of values taken by $\alpha(x)$ for $0 \leqq x \leqq l$.

Finally, the author states that if A is of class $(i\Omega)$, there is a triangular model $\mathbf{A}$ and a unitary transformation U

such that (1) U maps $H\ominus H_d$ on $\mathbf{H}\ominus\mathbf{H}_d$, and (2) $\mathbf{A}=UAU^*$ on the subspace $\mathbf{H}\ominus\mathbf{H}_d$. No proofs are given.
F. Smithies (Cambridge, England).

Livšic, M. S. On the resolvent of a linear nonsymmetric operator. Doklady Akad. Nauk SSSR (N.S.) **84**, 1131–1134 (1952). (Russian)

Continuing the investigation of an earlier paper [see the preceding review], the author gives an explicit expression for the resolvent of a triangular model operator in terms of the generalized Wronskian of a system of difference and differential equations associated with the operator. He then constructs a characteristic matrix function, which plays the same roles as that introduced in another paper [same Doklady **72**, 1013–1016 (1950); these Rev. **13**, 747] for quasi-Hermitian operators.

The following application to the theory of integral equations is then given. If $K(x, y)$ is a bounded kernel, whose real part is continuous and non-negative definite, then all its eigen-values (κ_n) lie in the half-plane $\Re(\kappa)\geqq 0$, and

$$\sum_n \Re(\kappa_n)\leqq \int_a^b \Re[K(x,x)]dx.$$

Equality holds here if and only if the system of principal functions of $K(x, y)$ is fundamental in the subspace generated by functions of the form

$$\int_a^b K(x,y)f(y)dy.$$

F. Smithies (Cambridge, England).

Lifšic, I. M. On a problem of the theory of perturbations connected with quantum statistics. Uspehi Matem. Nauk (N.S.) **7**, no. 1(47), 171–180 (1952). (Russian)

Continuing his work begun in previous articles [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **17**, 1017–1025, 1076–1089 (1947); these Rev. **9**, 358] the author obtains formulas for the case of an Hermitian operator with continuous spectrum subjected to a degenerate perturbation. The results are similar to the previous ones and are too complicated to cite here. There is a long section on applications.
B. Crabtree (Durham, N. H.).

Lehto, Olli. Some remarks on the kernel function in Hilbert function space. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **109**, 6 pp. (1952).

Let H be a Hilbert space whose elements f are real or complex functions defined in a domain D and let H possess a kernel function, i.e., a function $K_P \in H$, defined for every $P\in D$, such that $f(P)=(f, K_P)$. By two skillfully chosen examples, the author demonstrates the following two properties of the norm $\|K_P\|$: (a) a kernel K_P can exist without $\|K_P\|$ being bounded in every closed subdomain of D; (b) $\|K_P\|$ can be discontinuous although all functions of H are continuous.
Z. Nehari (St. Louis, Mo.).

Schäfke, Friedrich Wilhelm. Über Eigenwertprobleme mit zwei Parametern. Math. Nachr. **6**, 109–124 (1951).

In the eigen-value problem $Fy+\lambda y=0$, $\|y\|=1$, let F denote a Hermitean operator on a vector space with inner product, and let there be a discrete spectrum λ_r, $(r=0, \pm 1, \pm 2, \cdots)$. Then under certain conditions the λ-eigen-values of the perturbed problem $Fy+\lambda y+\mu Gy=0$, $\|y\|=1$, G being a bounded operator, will constitute a set of functions $\lambda_r(\mu)$ of μ. The author studies this dependence for complex μ, previous investigations having been mainly in the real field, notably B. de Sz. Nagy [Comment. Math. Helv. **19**, 347–366 (1947); these Rev. **8**, 589]; references are also given to earlier work of Rellich. Under various sets of supplementary hypotheses there are found lower bounds for the radii of convergence of the expansion

$$\lambda_m(\mu)=\lambda_m+\mu\lambda_{m1}+\mu^2\lambda_{m2}+\cdots$$

and upper bounds for the λ_{mp}, sharper than previous results. The proofs use complex-variable methods. Also considered is the case in which the eigen-value problem is non-linear in μ. The results are tested by application to the Mathieu and spheroidal differential equations. *F. V. Atkinson.*

Rothe, E. H. Leray-Schauder index and Morse type numbers in Hilbert space. Ann. of Math. (2) **55**, 433–467 (1952).

In a (not necessarily separable) Hilbert space let the scalar $i(x)$ be defined for some open ball U about θ by $i(x)=(2q)^{-1}[x,x]^q+l(x)$ where q is a positive integer and the bracket implies the scalar product. Let $G(x)=\operatorname{grad} l(x)$ be completely continuous and suppose θ is an isolated critical point of $g(x)=\operatorname{grad} i(x)$ in U. Let $K=\{i(x)\leqq 0\}\cap U$ and let $K'=\{i(x)\leqq 0\}\cap U-\theta$. Define m_r as $H_r(K, K')$ where the singular homology groups are taken. Denote the Leray-Schauder degree of $g(x)$ at θ by j. The author extends his earlier work and establishes the Morse relation $j=\sum(-1)^r m_r$ under rather natural regularity conditions (which he shows can be realized). The method of proof is to approximate by projection in E_n the n-dimensional Euclidean space. This is the Leray-Schauder method for j and is straightforward when mappings are layer mappings. The author gets to the layer mapping situation after extended argument. It is shown that $K_n(=K\cap E_n)$ and K_n' are deformation retracts of K and K' whence $m_r=H_r(K_n, K_n')$. Let $K(S)=K\cap S$, where S is a sphere about θ of sufficiently small radius. Then $K(S)$ is a deformation retract of K' and by an exactness argument, $H_r(K, K')\approx H_{r-1}(K(S))$, $r\geqq 2$. For the $K(S)$ groups it is possible to show that $i(x)$ may be replaced by a layer scalar, $s(x)$, yielding the same homology groups and for such a scalar the projection approximation argument gains the conclusion from the observation that the Morse relation is valid for E_n [E. Rothe, Math. Nachr. **4**, 12–27 (1951); these Rev. **12**, 720]. *D. G. Bourgin.*

Baluev, A. N. On the abstract theory of Čaplygin's method. Doklady Akad. Nauk SSSR (N.S.) **83**, 781–784 (1952). (Russian)

Let X and Y be linear subsets of partially ordered linear spaces $\hat{X}$ and $\hat{Y}$, respectively, and let $P(x)$ transform X into Y. The author generalizes Čaplygin's approximate method of solution of differential equations to equations of the form $P(x)=0$ with the following theorem. Let $u_0, v_0 \in X$, and let Γ_1, Γ_2 be additive homogeneous transformations of X into Y satisfying the following conditions: 1) $u_0\leqq v_0$, $P(u_0)\leqq 0\leqq P(v_0)$; 2) Γ_1 and Γ_2 have positive inverses $\Gamma_1^{-1}\geqq 0$, $\Gamma_2^{-1}\geqq 0$; 3) if $u_0\leqq x\leqq v_0$, then

$$P(v_0)+\Gamma_2(x-v_0)\leqq P(x)\leqq P(u_0)+\Gamma_1(x-u_0).$$

Then the elements u_1, v_1 determined by $u_1=u_0-\Gamma_1^{-1}P(u_0)$, $v_1=v_0-\Gamma_2^{-1}P(v_0)$ satisfy the inequalities: $u_0\leqq u_1\leqq v_1\leqq v_0$, $P(u_1)\leqq 0\leqq P(v_1)$. Moreover, for each x^* satisfying $P(x^*)=0$ and $u_0\leqq x^*\leqq v_0$, also $u_1\leqq x^*\leqq v_1$ holds. The author also states stronger versions of this theorem and gives an applica-

tion to differential equations of the form

$$x''-a(t)x'-f(x,t)=0, \quad x(t_0)=x(t_1)=0.$$

J. V. Wehausen (Providence, R. I.).

Kondô, Motokiti. La structure d'un flot topologique. I. Proc. Japan Acad. 25, no. 7, 1–10 (1949).

A topological flow is a pair (Ω, G) where Ω is a topological space and G is a group of homeomorphisms of Ω; it is topologically ergodic if, for any open O and P in Ω, gO, for some $g \in G$, intersects P. A complex-valued continuous function f on Ω is almost periodic if a certain family of translates is conditionally compact, and then, if the flow is ergodic, the mean value on g of $f(g\omega)$ is independent of ω and will be denoted by Mf. A scalar product is defined on the almost periodic functions by $(f,g)=M(f\bar{g})$, and then the group G has a regular representation by unitary operators on the completion of this. Analogues of the usual representation theory for compact groups are then obtained.

W. Ambrose (Cambridge, Mass.).

Theory of Probability

Britzelmayr, Wilhelm. Logisch-philosophische Bemerkungen zur Axiomatik der Wahrscheinlichkeitslehre. Miteilungsblatt Math. Statist. 4, 167–172 (1952).

Basu, D. On the limit points of relative frequencies. Sankhyā 11, 379–382 (1951).

The author shows that the condition of quasi-limit which was proposed by J. Singh [Sankhyā 7, 257–262 (1946); these Rev. 8, 35] in order to weaken the v. Mises condition of limiting frequency is in fact equivalent to the v. Mises condition. *A. H. Copeland, Sr.* (Ann Arbor, Mich.).

Domb, C. On the use of a random parameter in combinatorial problems. Proc. Phys. Soc. Sect. A. 65, 305–309 (1952).

The classical problem of finding the probability $p(m,r)$ of having m cells occupied if r balls are placed at random in n cells [cf., e.g., Feller, Probability theory, Wiley, New York, 1950, p. 69; these Rev. 12, 424], is considered in a formulation relating to counters and ascribed to Schrödinger. The calculation is simplified by considering r as a random variable with a Poisson distribution. If $P(m;\lambda)$ is the corresponding probability then $e^\lambda P(m,\lambda)=\sum \lambda^r p(m,r)/r!$. By the same method the author calculates the probability that m cells contain exactly k balls each. [Formulas (9) appear to be in error which, however, is of no consequence for the paper.] Finally, random divisions of the interval are treated by the same method. *W. Feller* (Princeton, N. J.).

Gardner, A. Greenwood's "problem of intervals": An exact solution for $N=3$. J. Roy. Statist. Soc. Ser. B. 14, 135–139 (1952).

Using a geometric method, the author obtains the distribution of the sum of squares of the four subintervals formed by three points selected at random in some interval.

G. E. Noether (Boston, Mass.).

Lévy, Paul. Sur une classe de lois de probabilité indécomposables. C. R. Acad. Sci. Paris 235, 489–491 (1952).

Let i_1 and i_2 be two intervals of lengths l_1 and l_2, respectively, which are less than the distance between i_1 and i_2 and let l_1/l_2 be irrational. Let c_1 and c_2 be two positive constants such that $c_1l_1+c_2l_2=1$. Define $f(x)=c_h$ for x in i_h $(h=1,2)$ and $=0$ elsewhere. Then $f(x)$ is the density function of an indecomposable distribution function $F(x)$ for which there exists a finite a such that $F(-a)=0$ and $F(a)=1$. This answers a question raised by Cramér [Ann. Math. Statistics 18, 165–193 (1947); these Rev. 8, 591]. The proof depends on the Lemma: two uniform distributions on intervals of length l_1 and l_2 are co-prime (in the sense of decomposition of distributions) if and only if l_1/l_2 is irrational. The lemma is a consequence of a more general statement about characteristic functions which are entire functions of order one. Finally, a theorem is proved which includes the example above as a special case.

K. L. Chung (Ithaca, N. Y.).

Prohorov, Yu. V. A local theorem for densities. Doklady Akad. Nauk SSSR (N.S.) 83, 797–800 (1952). (Russian)

Let $\xi_1, \xi_2, \cdots$ be mutually independent random variables with a common distribution, and let s_n be the sum of the first n. Let ϕ be the normal density, with mean 0 and variance 1. Then (1) the derivative of the distribution function of $s_n/n^{1/2}$ converges to ϕ in the mean of order 1 if and only if ξ_1 has mean 0 and variance 1 and if the distribution function of s_n is not singular for some n. More generally, (2) there is a choice of constants A_n, B_n such that the derivative of the distribution function of $(s_n-A_n)/B_n$ converges to a specified stable density in the mean of order 1 if and only if there is convergence of the corresponding distribution functions to the corresponding stable distribution function and if the distribution function of s_n is not singular for some n. (3) Suppose that ξ_1 has a finite moment of order $k\geqq 3$. Then under the hypotheses of (1) a function q_n is specified such that, if p_n is the derivative of the distribution function of $s_n/n^{1/2}$, $\int_{-\infty}^{\infty}|p_n-q_n|\,dx=o((\ln n)^{1/2}n^{(k-2)/2})$. The proof of (1) is sketched. *J. L. Doob* (Urbana, Ill.).

González Dominguez, Alberto, and Scarfiello, Roque. Limit theorems for products of random variables. Univ. Buenos Aires. Contrib. Ci. Ser. A. 1, 1–22 (1950). (Spanish. English summary)

It is noted that, since if $x_1, x_2, \cdots$ are independent, so are $y_1, y_2, \cdots$, where $y_i=\log x_i$, every limit theorem about sums of independent random variables can be stated as a limit theorem about products of independent random variables. Several examples are given. *D. Blackwell.*

Meizler, D. G. On a problem of B. V. Gnedenko. Ukrain. Mat. Žurnal 1, no. 2, 67–84 (1949). (Russian)

Let G be the class of distributions each of which can be obtained as the limiting distribution of X_n/a_n, where X_n is the maximum of the first n terms of a sequence of mutually independent random variables with distribution functions $F_1, F_2, \cdots$, and $a_1, a_2, \cdots$ are positive constants. It is supposed that, for each x, $\lim_{n\to\infty} F_i(a_nx)=1$ uniformly in $i\leqq n$. Let G^+ [G^-] be the class of G distributions confined to the positive [negative] half-lines. The following theorems are proved. (1) $\Phi \in G^+$ if and only if, for every number a with $0<a<1$, there is a non-decreasing function ϕ_a such that, for all x, $\Phi(x)=\Phi(x/a)\phi_a(x)$. Thus G is the class of distributions on the positive half-line which are limit laws of normalized sums of sequences of mutually independent random variables. (2) $G=G^++G^-$. (3) A G^+ distribution function has at most one discontinuity. If there is one at x_0, then either $x_0=0$ and the distribution is concentrated at the origin, or $x_0\neq 0$ and the distribution is confined to the

interval $[x_0, \infty)$. (4) In any interval of constancy, a G distribution function is identically 0 or 1. (5) A limit of G^+ $[G^-]$ distributions is of the same type. (6) If x is a positive random variable, then the distribution of x lies in G^+ if and only if that of $-1/x$ lies in G^-.

For the most general treatment of the special case in which F_n does not depend on n see Gnedenko [Ann. of Math. 44, 423–453 (1943); these Rev. 5, 41]. For a treatment of cases in which X_n/a_n above is replaced by $(X_n-b_n)/a_n$ see Meĭzler [Dopovidi Akad. Nauk Ukrain. RSR 1950, 3–10; these Rev. 13, 663]. Juncosa has found conditions on F_i sufficient to get various limiting Φ [Duke Math. J. 16, 609–618 (1949); these Rev. 11, 375]. *J. L. Doob.*

Prohorov, Yu. V. Some refinements of Lyapunov's theorem. Izvestiya Akad. Nauk SSSR. Ser. Mat. 16, 281–292 (1952). (Russian)

Let F_n be the distribution function of the sum, divided by $n^{1/2}$, of n mutually independent random variables with a common distribution function F of mean 0, variance 1, having finite absolute moment of order $\nu \geqq 3$. Let Φ be the normal distribution function with mean 0 and variance 1. (a) Suppose that F can be put in the form $F=pG_1+qG_2$, where $p>0$, $q>0$, $p+q=1$, and the G_1 distribution is concentrated at a finite number m of points on which a number theoretic condition (A) is imposed making the G_1 distribution not lattice-like. Then F_n is expressed in the usual asymptotic form of the central limit theorem, involving Φ and its derivatives, up to terms of order $n^{-\rho/2}$, $\rho=\min(\nu-2, m-2)$, with remainder

$$o(n^{-(\nu-2)/2}+(\log n)^{cm+d}/n^{(m-1)/2}).$$

Here c, d are involved in the condition (A). It is shown that this form of the remainder cannot be essentially improved. Let ψ be any function for which $\psi(n)=o(n^{-1/2})$. Examples of functions F in which (i) F is singular and continuous and (ii) F increases only in jumps, with the differences of the jump points commensurable, and having moments of all orders, are given such that, for any sequence of functions $\{G_n\}$ with uniformly bounded derivatives,

$$\limsup_{n\to\infty} \frac{\sup_x |F_n(x)-G_n(x)|}{\psi(n)}=\infty.$$

Example (i) is due to Kolmogorov. *J. L. Doob.*

Sverdrup, Erling. The limit distribution of a continuous function of random variables. Skand. Aktuarietidskr. 35, 1–10 (1952).

Let $\{F_n, n\geqq 1\}$ be a sequence of distribution functions in q dimensions, converging to the distribution function F at the continuity points of F. Suppose that the F distribution is confined to a closed set S, and that, for any ϵ-neighborhood of S, the F_n probability of being outside the neighborhood goes to 0 with $1/n$. Then, if g is a function defined on q-space, and continuous on S, it is shown that $\int e^{itg}dF_n \to \int e^{itg}dF$, so that, in the usual sense, the distribution assigned to g by F_n converges to that assigned to g by F. The novelty of the result lies in taking S rather than q-space as the domain of continuity of g. *J. L. Doob* (Urbana, Ill.).

Siraždinov, S. H. Refinement of limiting theorems for stationary Markov chains. Doklady Akad. Nauk SSSR (N.S.) 84, 1143–1146 (1952). (Russian)

Given a Markov chain with s states, with stationary transition probabilities, let $\mathbf{m}_n$ be the vector whose s components are the numbers of times the system is in the various states, in the first n steps. It is supposed throughout that (A) the states form a single recurrent class of period 1, that is, with no cyclically moving subclasses. Let Φ be the normal distribution function with mean 0 and variance 1, let $\mathbf{h}$ be a vector in s dimensions, and let F_n be the distribution function of the random variable $\mathbf{m}_n\cdot\mathbf{h}$, normalized appropriately. Then if a certain limiting variance is positive, $|F_n(x)-\Phi(x)|<c/n^{1/2}$, where c depends only on h and the given transition probabilities. A local theorem for the probabilities of values of $\mathbf{m}_n\cdot\mathbf{h}$ is stated under the assumption that the components of $\mathbf{h}$ form an arithmetic progression. Finally, if (A) is strengthened slightly, an asymptotic expansion is found for $n^{(s-1)/2}\mathrm{P}\{\mathbf{m}_n=\mathbf{x}\}$. Proofs are sketched. *J. L. Doob* (Urbana, Ill.).

Itô, Seizô. Brownian motions in a topological group and in its covering group. Rend. Circ. Mat. Palermo (2) 1, 40–48 (1952).

A Brownian motion on a topological group G is a Markov process whose random variables take on values in G, whose sample functions are almost all continuous, and which has (left) spatially invariant stationary transition probabilities. The author assumes that G satisfies the second countability axiom, is arcwise connected, and locally simply connected. The following theorem is proved. (A) If G^* is a covering group of G, then any Brownian movement on G, starting at the identity, induces a Brownian movement on G^* starting at the identity, by way of the natural path transformation. (B) Conversely, a Brownian motion on G^*, starting at the identity maps into one on G by way of the natural homomorphism, and the latter maps back into the original one by the map of (A). The transition probabilities of the two corresponding Brownian motions are related in the obvious way. These results generalize the case considered by Lévy [Bull. Soc. Math. France 67, 1–41 (1939); these Rev. 1, 62] who compared Brownian motions on circle perimeter and line. *J. L. Doob* (Urbana, Ill.).

Grenander, Ulf. On empirical spectral analysis of stochastic processes. Ark. Mat. 1, 503–531 (1952).

Let $\{x(t), -\infty<t<\infty\}$ be a stochastic process which is stationary in the wide sense. Let (a, b) be an interval, and let M be the class of random variables $\sum_1^n c_i x(t_i)$, $\sum_1^n c_i=1$, $t_i \epsilon (a, b)$, and of mean limits of these random variables. Then for each t there is an $x^*(t)$ in M closest (root mean square distance) to $x(t)$. It is shown that $m^*=\text{l.i.m.}_{A\to\infty}(2A)^{-1}\int_{-A}^{A}x^*(t)dt$ is the closest element of M to $E\{x(t)\}$ (supposed independent of t), so that m^* is the minimum variance unbiased estimate of $E\{x(t)\}$ based on $x(t)$ for $t \epsilon (a, b)$. Under further hypotheses m^* is related to predictions of $x(t)$ outside (a, b). Now suppose that t above is integer-valued and that the x_n process is Gaussian, with spectral density f. The distribution of the periodogram I_N, based on a sample of $x_{-N}, \cdots, x_N$, and of weighted averages $\int_{-\pi}^{\pi} I_N(s)dW(s)$, are discussed as estimates of f at specified values of the argument. *J. L. Doob* (Urbana, Ill.).

Grenander, Ulf. On Toeplitz forms and stationary processes. Ark. Mat. 1, 555–571 (1952).

The author summarizes the main properties of Toeplitz forms, and shows how many of the important theorems of prediction and estimation theory for stationary stochastic processes can be derived from these properties. *J. L. Doob* (Urbana, Ill.).

Miller, George A. Finite Markov processes in psychology. Psychometrika 17, 149–167 (1952).

Finite Markov processes are reviewed and considered for their usefulness in the description of behavioral data. (From the author's abstract.) *K. L. Chung* (Ithaca, N. Y.).

Kosten, L. On the accuracy of measurements of probabilities of delay and of expected times of delay in telecommunication systems. II. Estimates of average times of delay. Appl. Sci. Research B. 2, 401–415 (1952).

The expectation of waiting-time in delayed call systems can be estimated from the average number of waiting calls during a certain time interval. Another possibility to obtain the estimate is offered by averaging this number over all moments at which new calls appear. The latter estimate may be performed in two ways. The accuracies of the three estimates are computed and compared.

Author's summary.

Pollaczek, Félix. Sur la répartition des périodes d'occupation ininterrompue d'un guichet. C. R. Acad. Sci. Paris 234, 2042–2044 (1952).

This paper is concerned with a waiting line for which there is a given probability distribution for the time between arrivals of successive customers and a second distribution for the time required by a customer to transact his business. The author gives a method for computing the distribution function for the length of the interval between successive times when the line is free when there are exactly n customers in the line between free times.

A. H. Copeland, Sr. (Ann Arbor, Mich.).

Jacot, Henri. Théorie de la prévision et du filtrage des séries aléatoires stationnaires selon Norbert Wiener. Ann. Télécommun. 7, 241–249, 297–303, 325–335 (1952).

Expository paper.

Mathematical Statistics

✱Hald, A. Statistical theory with engineering applications. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1952. xii+783 pp. $9.00.

This book covers a wide range of topics. Statistical theory and practice have been welded to produce an original, unified, and scholarly text of unusual clarity and rigor. It is highly recommended both as a text and a reference. Theorems are systematically introduced and used over and over again. Diagrams and charts form an integral part of the text. All of this is done within the limitations of the usual calculus course. When proofs are beyond these limitations, the results are stated precisely with reference to the original source. Even recent results have been skillfully incorporated into the text with frequent simplifications of the original proofs. This book should correct a common impression that the only statistics worthy of an engineer's attention is the theory of statistical quality control. Some salient features are the transformations of probability distributions to approximately normal ones, the abundance of practical problems, emphasis on the logic of statistical quality control throughout, the thorough treatment of the analysis of variance, the design of experiments, regression theory in general, the interpretation of results, and the power function of statistical tests. A complete list of topics covered would be too lengthy for the present review. Omitted is any discussion of the moment generating and the characteristic functions. A lengthier section on time series and some discussion of the statistical theory of noise including autocorrelation functions and power spectra would be useful for electrical engineers. These remarks are not intended to detract from the overall excellence of the book. A somewhat more dubious matter is the approximate formula for the mean and variance of functions of x given on pages 246–251. For instance it is stated that the coefficient of variation of $1/x$ approximately equals the coefficient of variation of x. If x is normally distributed, the result is meaningless as the moments of $1/x$ do not exist, no matter how small the coefficient of variation of x, the condition given by the author. In the particular applications, the author has used these formulas properly. On pages 301–303 is given an ingenious derivation of the approximate distribution of the sample coefficient of variation c from a normal population. While this approximation is excellent, it seems that the result of McKay [J. Roy. Statist. Soc. 95, 695–698 (1932)], if γ indicates the population coefficient of variation, to the effect that $(\gamma^{-2}+1)fc^2/(1+c^2)$ is distributed as χ^2 with f degrees of freedom, is somewhat more accurate and simpler to use. The exact distribution of c^2 was found by P. C. Tang [Statist. Res. Mem. London 2, 126–149 (1938), p. 140]. It is regrettable that there are no problems for students to work. *L. A. Aroian* (Culver City, Calif.).

✱Hald, A. Statistical tables and formulas. John Wiley and Sons, Inc., New York; Chapman and Hall, Limited, London, 1952. 97 pp. $2.50.

This is a welcome addition to existing statistical tables. The first part is devoted to a collection of statistical formulas and uses of the tables in the notation of the author's book [see preceding review]. In most cases linear interpolation suffices. Beneath each table is an example of its use. Particularly noteworthy are the tables of fractiles (percentage points) of Student's t distribution, the χ^2 distribution, the v^2 (analysis of variance) distribution, and the tables of confidence limits for the binomial. Tables I, II, and III provide ordinates of the normal curve, the cumulative normal distribution, and probits of the normal distribution respectively.

Table IV. Fractiles of the t distribution t_P to 3 decimal places for f (degrees of freedom) $=1$ (1) 30, 40, 50, 60, 80, 100, 200, 500, ∞ with $P=.60$, .70, .80, .90, .95, .975, .99, .995, .999, .9995. Table V. Fractiles of the χ^2 distribution given to 3 or 4 significant figures, $f=1$ (1) 100, $P=.0005$, .001, .005, .01, .025, .05, .10 (.10) .90, .95, .975, .99, .995, .999, .9995. Table VI. Fractiles of the χ^2/f distribution, given to 4 decimals, $f=1$ (1) 100 (5) 200 (10) 300 (50) 1000 (1000) 5000 (5000) 10,000. This table is new.

Table VII consists of 9 parts and is perhaps the most extensive table of the analysis of variance, $v^2(f_1, f_2)$, distribution (ordinarily designated by F). $\mathrm{VII}_{1,2,3}$ provide 50%, 70%, and 90% fractiles to 3 significant figures respectively, f_1 and $f_2=1$ (1) 10, 15, 20, 30, 50, 100, 200, 500, ∞. $\mathrm{VII}_{4,5,6,7}$ provide 95%, 97.5%, 99%, and 99.5% fractiles to 3 significant figures respectively for $f_1=1$ (1) 20 (2) 30 (5) 50, 60, 80, 100, 200, 500, ∞, and for f_2 the same values and in addition 1000. $\mathrm{VII}_{8,9}$ provide 99.9% and 99.95% fractiles to 3 significant figures respectively for f_1 and $f_2=1$ (1) 10, 15, 20, 30, 50, 100, 200, 500 and ∞. Table VII_9 is new.

Table VIII gives fractiles to 3 significant figures for the distribution of the range and factors useful in quality control

work for samples of size 2 (1) 20. Tables IX and X are useful in problems concerned with the one-sided truncated and one-sided censored normal distribution. Tables XI and XII (new) give two-sided 95% and 99% confidence limits for the probability of success in the binomial distribution to 3 significant figures for practically all values of the relative frequency x/n by means of an ingenious arrangement in which $x=0$ (1) 20 (2) 30 (5) 50, 60, 80, 100, 200, 500, ∞ and $n-x$ the same values except 0 (which could also have been included). Table XII is that of 2 arcsin $x^{1/2}$, $x=0$ (.001) 1.

Table XIII. $\log_{10} n!$, $n=1$ (1) 1000. Table XIV. $\log_{10} \binom{n}{x}$, $n=1$ (1) 100, $x=1$ (1) 50. The book concludes with tables of squares (XV), square roots (XVI), reciprocals (XVIII), common logarithms (XIX), and a table of 15,000 random numbers (XX). *L. A. Aroian* (Culver City, Calif.).

Auxiliary table for Wilcoxon's two sample test. Mathematical Centre, Amsterdam, Rep. R132/S86. 35 pp. (1952).

These tables give values (1D) of $\mu=mn/2$, and (2D) of $\sigma=(mn(m+n+1)/12)^{\frac{1}{2}}$, $m=1(1)100$, $n=11(1)100$; also of t^3-t, $t=1(1)100$. These provide means of computing readily the well-known test of F. Wilcoxon [Biometrics Bull. 1, 80–83 (1945)] for the hypothesis that two samples of m and n members respectively are drawn from the same population. The general formula for σ^2 was derived by J. Hemelrijk and here simplified by T. J. Terpstra. The first two tables suffice, save for a minor correction (using t^3-t) due to ties.

A. A. Bennett (Providence, R. I.).

Sato, R. Errata. Ann. Inst. Statist. Math., Tokyo 3, 127–128 (1952).

Errata to two papers in same Ann. 2, 91–124 (1951); 3, 45–46 (1951); these Rev. 13, 52, 571.

Fréchet, Maurice. Sur les tableaux de corrélation dont les marges sont données. Ann. Univ. Lyon. Sect. A. (3) 14, 53–77 (1951).

Let $F(x)$ and $G(x)$ be distribution functions. It is shown that

$$H_0(x,y)=\max[F(x)+G(y)-1,0] \text{ and } H_1=\min[F(x),G(y)]$$

are distribution functions with F and G as marginal distributions and that every distribution H with this property satisfies $H_0 \leqq H \leqq H_1$. A probabilistic interpretation of H_0 and H_1 is given. A similar problem is solved for frequency arrays where the entries are necessarily integers.

W. Feller (Princeton, N. J.).

Shah, A. B. Polynomial expansion of a cumulant. J. Indian Soc. Agric. Statistics 4, 91–93 (1952).

In a recent paper Wishart [Biometrika 36, 47–58 (1949); these Rev. 11, 528] has given a differential recursion formula for the cumulants of multinomial multivariate distributions. The object of the present note is to indicate a method of expanding the general cumulant of a distribution, satisfying a simple condition, in a polynomial of lower order cumulants. Examples of the result are applied to univariate and bivariate distributions. *L. A. Aroian* (Culver City, Calif.).

Toranzos, Fausto I. An asymmetric bell-shaped frequency curve. Ann. Math. Statistics 23, 467–469 (1952).

The author investigates the properties of the frequency curve $y=kx^c \exp\{-\frac{1}{2}a^2(x-b)^2\}$, $0 \leqq x \leqq \infty$.

L. A. Aroian (Culver City, Calif.).

Banerjee, D. P. On the moments of the multiple correlation coefficient in samples from normal population. J. Indian Soc. Agric. Statistics 4, 88–90 (1952).

The author finds the mth moment about the origin of the square and the first power of the multiple coefficient of correlation in samples from a normal population in terms of a generalized hypergeometric function. *L. A. Aroian.*

Goodman, Leo A. On the Poisson-Gamma distribution problem. Ann. Inst. Statist. Math., Tokyo 3, 123–125 (1952).

Let $X_1, X_2, \cdots$ be a sequence of non-negative independent random variables with the same continuous distribution function $F(x)$, and let N_x be defined by

$$N_x=\begin{cases}0 & \text{if } X_1>x\\ n & \text{if } \sum_{i=1}^{n} X_i \leqq x \text{ and } \sum_{i=1}^{n+1} X_i>x;\end{cases}$$

if N_x is distributed according to a generalized Poisson law with fixed $\alpha \leqq 2$ for every positive x, then $F(x)$ is a gamma type distribution. *L. A. Aroian* (Culver City, Calif.).

Vora, Shanti A. Bounds on the distribution of chi-square. Sankhyā 11, 365–378 (1951).

The author finds three bounds on the distribution of χ^2 where $\chi^2=\sum\{(v_j-Np_j)^2/Np_j\}$. The exact theorems are too lengthy to be given here. Given the hypothesis H that the data form a sample of N independent observations from a probability function $P(S)$ completely specified; let the space be divided into k disjoint sets $S_1, \cdots, S_k$, with $P(S_j)=p_j>0$, and $\sum_{j=1}^{k} p_j=1$. Let v_j be the observed frequency in the set S_j, $v_j \geqq 0$, $\sum v_j=N$. The joint distribution of the $v_1, \cdots, v_k$ is given, of course, by the multinomial distribution, and χ^2, as defined above, is used to test the hypothesis H. Essentially the paper treats results for finite N, while the usual χ^2 theory is correct as $N\to\infty$. *L. A. Aroian.*

Chapman, Douglas G. Sufficient statistics for "selected distributions". Univ. Washington Publ. Math. 3, 59–64 (1952).

Tukey's theorem [Ann. Math. Statistics 20, 309–311 (1949); these Rev. 10, 723] concerning sufficient statistics for a truncated or selected family of distributions is shown to be true for a wider class of families of distributions than that considered by Tukey. Additional theorems concerning the existence of complete minimal sufficient statistics and of unbiased estimates for families of distributions obtained by selection are also proved. *G. E. Noether.*

Rosenblatt, Murray. Remarks on a multivariate transformation. Ann. Math. Statistics 23, 470–472 (1952).

The author discusses a transformation which transforms any absolutely continuous k-variate distribution into the uniform distribution over a k-dimensional hypercube. He notes that the probability distributions of the Kolmogorov-Smirnov and von Mises statistics for measuring the agreement of sample and population distributions are not invariant with respect to the population distribution in the multivariate case. He proposes that data be subjected to the transformation before calculating the statistics mentioned, also before applying the chi-square test of goodness of fit.

S. W. Nash (Vancouver, B. C.).

Kôno, K. Note on the use of order statistics. Bull. Math. Statist. **4**, 33–35 (1950).

For the case when the joint distribution of x, y is symmetric about $(E[x], E[y])$ the author finds an unbiased estimate F of $E[x]$ by the following procedure. Draw λ random samples of n units each, observe x only on those units which carry the ith and $(n-i+1)$th of the ordered y-values of each sample $(n-i+1<i)$, and compute F as the average of the observed x-values. He derives the large sample variance V_0 of F for the case when x, y are normally correlated, obtains, for given total cost and given costs per observation on x and y, min (V_0) with respect to n and i, and gives a condition under which min (V_0) is smaller than the corresponding variance based on simple random sampling. [Reviewer's remark. There are some errors, which, however, do not influence the results.]

D. M. Sandelius (Uppsala).

Kôno, K. Note on the double sampling method. Bull. Math. Statist. **4**, 36–38 (1950).

The author compares Neyman's double sampling method [J. Amer. Statist. Assoc. **33**, 101–116 (1938)] with simple random sampling, assuming the stratification and estimation variates to be normally correlated.

D. M. Sandelius (Uppsala).

Murakami, M. Some considerations on the ratio and regression estimates. Bull. Math. Statist. **4**, 39–42 (1950).

The efficiencies of ratio, regression, and mean per sample unit estimates are compared, under cost considerations, for the cases of single and double sampling.

D. M. Sandelius (Uppsala).

K. N. M. A note on correlation between two unbiased estimators. Calcutta Statist. Assoc. Bull. **4**, 72–73 (1952).

Let ρ be the correlation of two unbiased estimates with efficiencies e_1 and e_2, respectively. Then in the regular estimation case as defined by Cramér [Mathematical methods of statistics, Princeton Univ. Press, 1946, p. 479; these Rev. **8**, 39] the following inequalities hold:

$$(e_1e_2)^{1/2}-[(1-e_1)(1-e_2)]^{1/2}\leqq\rho\leqq(e_1e_2)^{1/2}+[(1-e_1)(1-e_2)]^{1/2}.$$

G. E. Noether (Boston, Mass.).

Cox, D. R. A note on the sequential estimation of means. Proc. Cambridge Philos. Soc. **48**, 447–450 (1952).

A method is given for finding, at the end of a sequential sampling procedure, an almost unbiased estimate of the population mean. The author gives examples illustrating several applications of the method. *R. P. Peterson.*

Cochran, William G. The χ^2 test of goodness of fit. Ann. Math. Statistics **23**, 315–345 (1952).

The paper is expository. It emphasizes problems arising in the practical application of the chi square test and related tests. *W. Hoeffding* (Chapel Hill, N. C.).

Terry, Milton E. Some rank order tests which are most powerful against specific parametric alternatives. Ann. Math. Statistics **23**, 346–366 (1952).

A method of Hoeffding [Proc. Second Berkeley Symposium Math. Statist. Prob., 1950, pp. 83–92, Univ. of California Press, 1951; these Rev. **13**, 479] is applied to testing the hypothesis that N observations come from the same (unknown) population while under the alternative hypothesis the observations come from normal distributions with different means. In particular, the two-sample problem with normal alternatives is considered. If m and n are the respective sample sizes, tables for the application of the optimum rank order test are provided for $m+n\leqq 10$. Asymptotic results as well as a comparison with the Mann and Whitney U-Test are given. *G. E. Noether.*

Barankin, Edward W. On systems of linear equations, with applications to linear programming and the theory of tests of statistical hypotheses. Univ. California Publ. Statist. **1**, 161–214 (1951).

For a given set $u_0, u_1, \cdots, u_k$ of vectors and a set $c_1, \cdots, c_k$ of constants, the author considers the problem of maximizing and minimizing the linear function $L(x)=\sum_1^k c_ix_i$ over the set X of all $x=(x_1, \cdots, x_k)$ with $x_i\geqq 0$ and $u_0=\sum_1^k x_iu_i$. If X is not empty and $\alpha_0=\sup_{x\in X} L(x)$ and $\beta_0=\inf_{x\in X} L(x)$ are finite, there is a unique finite set $x_1, \cdots, x_n$ in X such that each x_i has a minimal set of positive coordinates; every $x\in X$ has a representation $x=\gamma+\sum_1^r p_kx_k$, where $\sum_1^k c_i\gamma_i=0$ and $\sum_1^r p_k=1$. If $\sum_1^k x_iu_i=0$ and $x_i\geqq 0$ imply $x_i=0$ and if $c_i>0$, X is non-empty if and only if there is a positive M with

$$(u_0, v)\leqq M \max_{i=1,\cdots,k} (u_i, v) \tag{1}$$

for all v; moreover if X is non-empty, M satisfies (1) for all v if and only if $\beta_0\leqq M\leqq\alpha_0$. Under the hypotheses of the preceding sentence, further detailed results concerning the structure of X are obtained.

Among the applications to testing hypotheses is the following. If $m_1, \cdots, m_n$ are probability measures on a finite space Ω, a necessary condition that a vector $v=(v_1, \cdots, v_n)$ belong to R, the convex hull of the range of the vector measure $m=(m_1, \cdots, m_n)$, is that there exist a positive M with

$$\sum_{j=1}^n t_jv_j\leqq M \max_{\omega\in\Omega}\sum_{j=1}^n t_jm_j(\omega)$$

for all $t=(t_1, \cdots, t_n)$. *D. Blackwell* (Washington, D. C.).

Okamoto, Masashi. On a non-parametric test. Osaka Math. J. **4**, 77–85 (1952).

David [Biometrika **37**, 97–110 (1950); these Rev. **12**, 38] has proposed the following procedure for testing the hypothesis H_0 that a random sample of size N has come from a given continuous population. Divide the range of a single observation into n equally probable (according to H_0) intervals. Reject H_0 if the number v of intervals not containing any observations is too large. The present author shows that, under H_0, v is asymptotically normally distributed as N, $n\to\infty$ in a constant ratio. The test is also shown to be consistent and unbiased with respect to a wide class of alternatives. *G. E. Noether* (Boston, Mass.).

Roy, S. N. On a property of Bayes solutions in the Neyman-Pearson set-up. Calcutta Statist. Assoc. Bull. **4**, 67–71 (1952).

The author points out that what is formally a Bayes solution test of a hypothesis H_0 against a class of different alternatives H may also be regarded as a test which among the tests having identical power $\beta(H)$ minimizes the probability of an error of the first kind. In the author's opinion this is a more important property of Bayes solutions than that of admissibility, at least from a practical point of view.

G. E. Noether (Boston, Mass.).

Mourier, Edith. Tests de choix entre diverses lois de probabilité. Trabajos Estadística 2, 233–260 (1951). (French. Spanish summary)

The first part of this paper consists of known results, e.g., for a finite set of distributions, the set of likelihood ratios is a sufficient statistic; for testing a simple hypothesis against a simple alternative, every Bayes solution is a likelihood ratio test. In the second part, tests of a simple hypothesis against a simple alternative are considered in more detail, as follows. For a given a priori probability of each hypothesis, let P_n be the overall probability of error from a sample of size n. Using the central limit theorems, it is shown that $P_n \sim ce^{-un}n^{-1/2}$ as $n\to\infty$, where c, u are positive constants depending on the two distributions and c depends also on the given a priori probability. In view of this result, it is suggested that u is a useful measure of the "distance" between two distributions. For two normal distributions with means a_1, a_2, and variance 1,

$$u=\tfrac{1}{8}(a_1-a_2)^2 \min [(a_1-a_2)^2, 1].$$

D. Blackwell (Washington, D. C.).

Wünsche, Günther. Bemerkungen über nomographische Verfahren zur rationellen Sequentialtest-Planung. Mitteilungsblatt Math. Statist. 4, 268–276 (1 plate) (1952).

Dalenius, Tore. Eine einfache geometrische Veranschaulichung der Theorie des geschichteten Stichprobenverfahrens. Mitteilungsblatt Math. Statist. 4, 121–128 (1 plate) (1952).

Whittle, Peter. Some results in time series analysis. Skand. Aktuarietidskr. 35, 48–60 (1952).

The author summarizes and makes somewhat more precise the purely mathematical work of his thesis [Uppsala Univ., 1951; these Rev. 12, 726]. The proofs are only outlined. His results apparently apply to a stationary Gaussian process whose strictly deterministic component has spectral density whose Fourier expansion is the Laurent series on the unit circle perimeter of a function regular in a ring containing this perimeter. (This hypothesis on the spectral density is not stated in the paper.) *J. L. Doob.*

Ogawa, Junjiro. A remark on the efficiency of the designs of weighing experiments. Proc. Japan Acad. 27, 532–535 (1951).

The author proposes a new definition for the efficiency of a weighing design to replace one due to Kishen [Ann. Math. Statistics 16, 294–300 (1945); these Rev. 7, 133] in order to take into account the fact that the best estimates of the weights are not independent. *R. C. Bose.*

Nair, K. R. Analysis of partially balanced incomplete block designs illustrated on the simple square and rectangular lattices. Biometrics 8, 122–155 (1952).

This paper is an exposition of the analysis of the designs of the title. However explicit expressions are obtained for the combined intra and inter block analysis of p.b.i.b. designs with 4 associate classes. *H. B. Mann.*

TOPOLOGY

Taimanov, A. D. On quasi-components of disconnected sets. II. Mat. Sbornik N.S. 30(72), 465–482 (1952). (Russian)

For notation and terminology not explained here, see the review of part I [Mat. Sbornik 25(67) 367–386 (1949); these Rev. 11, 335]. The points of E are now divided into two classes. An $x \in E$ is of the first kind if for some neighborhood $U(x)$, every point $y \in U(x)$ has index $<\beta$, where β is a fixed ordinal $<\Omega$. Otherwise x is of the second kind. The set E_j of points of the second kind, which is obviously closed, is called the kernel of disconnectivity of E. The least ordinal number exceeding all indices of points of the first kind is called the index of the kernel E_j. Theorem: the space E_j contains in its relative topology no points of the first kind. It is shown that all ordinals $<\Omega$ can appear as indices of kernels, and by complicated constructions that there exist locally compact E and E of dimension 1 with arbitrary index α and non-void kernel. Dividing E into its components and 1-quasi-components, and introducing various topologies into the spaces whose points are these sets, the author studies the structure of these spaces in terms of the structure of E. *E. Hewitt* (Seattle, Wash.).

Arens, Richard. Extension of functions on fully normal spaces. Pacific J. Math. 2, 11–22 (1952).

A number of extension theorems are shown to hold in a wider setting than known previously. Among other results, Hausdorff's extension theorem [Fund. Math. 30, 40–47 (1938)] and Kakutani's simultaneous extension theorem [Jap. J. Math. 17, 1–4 (1940); these Rev. 2, 104], both proved originally for separable metric spaces, are here shown to hold for arbitrary metric spaces. A basic result due to Dugundji [Pacific J. Math. 1, 353–367 (1951); these Rev. 13, 373] is generalized to: any mapping of a closed subset of a fully normal space X into a complete convex metric subset of a convex topological linear space K can be extended to all of X. Also, a generalization of Tietze's theorem is proved, the statement of this generalization being obtained from the above by allowing X to be merely normal but K a compact convex subset of a normed linear space.

Two references are omitted from the list at the end of the paper: 13) J. W. Tukey, Convergence and uniformity in topology, Princeton, 1940; 14) J. V. Wehausen, Transformations in linear topological spaces, Duke Math. J. 4, 157–169 (1938). *E. G. Begle* (New Haven, Conn.).

Ellis, David. On separable metric spaces. Univ. Nac. Tucumán. Revista A. 8, 15–18 (1951).

Let $p_1, p_2, p_3, \cdots$ be a countable dense subset of a separable metric space M. Then each point q of M can be given a countable sequence of so-called metric coordinates, where the ith such coordinate is the distance from q to p_i. This short note is a discussion of a few of the elementary properties of these metric coordinates. *D. W. Hall.*

Wagner, K. Bemerkungen zur Dimension des Durchschnitts von Punktmengen. Arch. Math. 3, 79–82 (1952).

Let P be a polyhedron of dimension d, and I the closed unit interval on the real axis. Then for each x in I there exists a homeomorphic image P_x of P in the n-dimensional Euclidean space R_n $(0\leqq d\leqq n)$. For p in P, p_x denotes the image of p under this mapping. We say that the system of sets $(P_x)_{x\in I}$ is continuous in y at the point q of P provided that for each positive number e there exists a neighborhood G of q in P and a real number $h>0$ such that for each p in

G and each x in I with $|y-x|<h$ the distance between the two points p_x, p_y is smaller than ϵ. The system $(P_x)_{x\epsilon I}$ is continuous if it is continuous at each point q of P for each y in I. We say that $(P_x)_{x\epsilon I}$ is locally one-to-one provided that to each subinterval U of I and each d-dimensional simplex S which is a subset of P, there exists a subinterval V of U and a d-dimensional subsimplex T of S such that p_y and p_x are distinct points for every pair of distinct real numbers x, y of V and each pair of (not necessarily distinct) points p, q of T.

The paper consists of a proof of the following theorem and a short discussion of several of its special cases. Given a d-dimensional (finite or infinite) polyhedron P which gives rise to a continuous and locally one-to-one system of point sets $(P_x)_{x\epsilon I}$ in R_n $(0\leqq d\leqq n)$, then for each d-dimensional closed subset A of R_n the set M of all x in I such that P_x has an at most $(d-1)$-dimensional intersection with A is of the second category. An example is given to show that the conclusion of the theorem is as strong as possible.

D. W. Hall (College Park, Md.).

Nöbeling, Georg. Ein gemeinsamer Beweis für den Jordanschen Kurvensatz und zwei damit zusammenhängende Sätze. J. Reine Angew. Math. **188**, 22–39 (1950).

The theorems referred to in the title (in addition to the Jordan Curve Theorem) are: I) A domain (in the plane or the 2-sphere S^2) which has a non-empty connected boundary is homeomorphic to an open disc; II) if h is a topological mapping of a circle or closed linear interval of S^2 into S^2, then h can be extended to a homeomorphism of S^2 onto itself (called by the author the "Antoine extension theorem," although first proved, for the circle, by Schoenflies [Math. Ann. 62, 286–328 (1906)]). The method depends strongly on the concept of "convex envelope" (konvexe Hüllkurve); in the case of a plane, compact set M, this is the boundary of the intersection of all closed half-planes that contain M.

The Jordan Curve Theorem is given simple independent proofs for certain special cases in which the curve is essentially piecewise linear (although not necessarily finite), but the general case is made a corollary of II. In proving I, it is shown that if M is any continuum, then each component D of S^2-M is homeomorphic with an open disc; and that if, moreover, M is locally connected, then the homeomorphism can be extended to a continuous mapping of the closed disc onto $\bar{D}$ carrying the boundary of the disc onto the boundary of D. From the latter follows the Torhorst theorem that the boundary of a domain complementary to a locally connected plane continuum is itself locally connected, and certain well-known accessibility theorems. Thus the paper achieves a unified treatment of diverse theorems not usually associated with one another. *R. L. Wilder.*

Whyburn, G. T. On quasi-compact mappings. Duke Math. J. **19**, 445–446 (1952).

Assume that f is a map of X onto Y. Then f is quasi-compact if, when A is open and $A=f^{-1}f(A)$, then $f(A)$ is open. If $f(X_0)=Y$ and $f|X_0$ is quasi-compact, then f is quasi-compact. Retractions and open and closed maps are quasi-compact. Local connectedness is preserved under quasi-compact maps. *A. D. Wallace* (New Orleans, La.).

Titus, C. J., and Young, G. S. A Jacobian condition for interiority. Michigan Math. J. **1**, 89–94 (1952).

It is shown that if D is an open set in E^n and f is a mapping of class C^1 of D into E^n, then: (a) if the Jacobian $J(f)$ vanishes only on a compact set in D of dimension less than $n-1$, f is quasi-interior on D; (b) if f is light and $J(f)$ is non-negative (non-positive) in D, f is interior. Also if D is conditionally compact and f is continuous on $\bar{D}$, of class C^1 on D, and satisfies the condition in (a), then f can be factored into a monotone map m: $\bar{D}\to M$ with $m(D)$ open in M, followed by a light map l: $M\to E^n$ which is interior on $m(D)$. In the case of a function $f(z)$ having a continuous derivative in the domain D of the complex plane, the authors' results show that f is interior provided it is known that either (i) the zeros of $f'(z)$ form a zero-dimensional set or (ii) f is light and is not constant on any open subset of D.

G. T. Whyburn (Stanford, Calif.).

Bing, R. H. A homeomorphism between the 3-sphere and the sum of two solid horned spheres. Ann. of Math. (2) **56**, 354–362 (1952).

Let K be a horned-sphere of J. W. Alexander and D the complementary domain of K whose fundamental group does not vanish. Let H be the space obtained as the "union" of two copies of $K\cup D$, corresponding points of K being "identified". The author shows, solving a problem proposed by R. L. Wilder, that H is topologically a 3-sphere. From the construction of H it is clear that H admits a homeomorphism h of period 2 with K as fixed-point set, and which interchanges points of the complementary domains of K in H. The existence of such an h leads to a negative answer to a question of Deane Montgomery: Is a compact group acting on Euclidean space equivalent to an Euclidean group? Related (if not indeed equivalent) questions have been considered by S. Eilenberg, P. A. Smith, and L. Zippin. The author constructs (a related example was given by L. Antoine) a Cantor set C in 3-space whose complementary domain has a non-vanishing fundamental group but such that each two points of C can be separated by a topological 2-sphere in the complement of C. The example of Antoine has the property that no pair of points can be so separated. Interesting problems are implicit and explicit in the paper. The arguments are of a set-theoretic character.

A. D. Wallace (New Orleans, La.).

Bing, R. H. Partitioning continuous curves. Bull. Amer. Math. Soc. **58**, 536–556 (1952).

The author gives a full discussion, mostly informal, of his theory of partitionings of continuous curves, with various applications. A continuous curve M is a compact metric space which is connected and locally connected. A partitioning of M is a finite collection G of disjoint connected open sets g, the sum of whose closures is M. If each g of G has property S, then G is an S-partitioning; here g has property S if, for each $\epsilon>0$, g is the sum of a finite collection of connected sets each of which has diameter $<\epsilon$. If each g of G is uniformly locally connected, and is the interior of its own closure, and if also the interior of the closure of the sum of any two elements of G is uniformly locally connected, then G is a brick partitioning. A sequence $G_1, G_2, \cdots$ of partitionings of M is descending if (1) for each i, G_{i+1} is a refinement of G_i, and (2) the maximum diameter of the elements of G_i approaches 0 as i goes to infinity. If G_{i+1} is a refinement of G_i, and $g\,\epsilon\,G_i$, then $C(g)$ is the closure of the sum of all elements of G_{i+1} whose closures lie in g. If each such $C(g)$ is connected, and intersects the closure of every element of G_{i+1} that lies in g, then G_{i+1} is a core refinement of G_i. The basic existence theorems are the following. (I) Every continuous curve has a decreasing sequence of brick partitionings. (II) Every continuous curve M has a descending

sequence $G_1, G_2, \cdots$ of S-partitionings, such that for each i, G_{i+1} is a core refinement of G_i.

The principal applications are to the Menger convexification problem (that of showing that every continuous curve can be given a convex metric which preserves the given topology) and to the solution, in generalized form, of the Kline sphere characterization problem (that of showing that a continuous curve which is separated by no pair of points, but by every simple closed curve, is a 2-sphere.)

Various extensions of Theorems (I) and (II) are mentioned. For example, the author announces that if M is n-dimensional, then M has a descending sequence $G_1, G_2, \cdots$ of partitionings, such that (1) each g of each G_i is the interior of its closure, and (2) each G_i is dimensionally regular, in the sense that the intersection of the boundaries of any j different elements of G_i is a set of dimension $\leqq n+1-j$ (for $j \leqq n+1$). In conclusion the author cites various related results due to himself, E. E. Floyd, and Garth Thomas.

E. E. Moise (Ann Arbor, Mich.).

Noguchi, Hiroshi. A note on absolute neighborhood retracts. Tôhoku Math. J. (2) 4, 93–95 (1952).

There are two possible definitions for an ANR(=absolute neighborhood retract). According to one definition, a space X is an ANR if it is a neighborhood retract of any space of which it is a closed subset. In the second definition, the restriction that X be a closed subset is removed. In this note it is shown that in locally compact spaces these two definitions are equivalent. *E. G. Begle.*

Kinoshita, Shin'ichi. On essential components of the set of fixed points. Osaka Math. J. 4, 19–22 (1952).

An essential component of a mapping f of a compactum X into itself is a component C of the set of fixed points of f with the property that for any open set U containing C there is a positive number δ such that if g is a mapping of X into itself with $|f-g|<\delta$, then g has a fixed point in U. It is shown that if X is an absolute retract, then every mapping of X into itself has an essential component. For absolute neighborhood retracts, the same is true for mappings homotopic to a constant mapping. *E. G. Begle.*

Ganea, Tudor. Covering spaces of topological products. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 199–205 (1950). (Romanian. Russian and French summaries)

A connected, locally connected Hausdorff space E will be called here a C-space if it has a covering space $\tilde{E}$ that is maximal in the sense that $\tilde{E}$ has no covering other than itself. The group of covering transformations of $\tilde{E}$ will be denoted by $C(E)$. The author proves that the Cartesian product E of a family $\{E_\alpha\}$ is a C-space if and only if each E_α is a C-space and $C(E_\alpha)=1$ for almost all α, and that, then, $C(E)=1$ if and only if $C(E_\alpha)=1$ for all α. Corollaries: A compact ANR is a C-space; if E is a compact AR, then $C(E)=1$; S_1^∞ is not a C-space. *R. H. Fox.*

Ganea, Tudor. Simply connected spaces. Fund. Math. 38, 179–203 (1951).

The author is concerned with the group $C(E)$ of covering transformations of the maximal covering $\tilde{E}$ of a C-space E [see review above for notations], and with conditions under which $C(E)=1$, or, what is the same thing, $\tilde{E}=E$. The principal results are the following: If X and Y are quasi-homeomorphic (i.e., there exist ϵ-mappings of X on Y for every ϵ, and conversely) paracompact C-spaces, then each of the groups $C(X)$, $C(Y)$ contains a retract of the other, and $C(X)=1$ if and only if $C(Y)=1$. Let X and Y be C-spaces and φ a monotone map of X on Y. Then φ induces a homomorphism of $C(X)$ onto $C(Y)$; in particular, $C(Y)=1$ if $C(X)=1$. This homomorphism is an isomorphism under certain additional conditions. Let φ be a continuous map of a compact C-space X upon a compact C-space Y. If $C(Y)=1$ and, for each $y \in Y$, $\varphi^{-1}(y)$ is a C-space and $C(\varphi^{-1}(y))=1$, then $C(X)=1$. Let $F_1 \supset F_2 \supset \cdots$ be a sequence of compact C-spaces such that $\bigcap F_i$ is locally connected. The inverse limit of the sequence $C(F_1) \leftarrow C(F_2) \leftarrow \cdots$ contains $C(F)$ as a retract. In particular, $C(F)=1$ if all $C(F_i)=1$. There is a Peano space $\mathfrak{A}$ in 3-space (originally constructed by Borsuk for a different purpose) such that (i) $\mathfrak{A}$ is the intersection of a sequence $F_1 \supset F_2 \supset \cdots$ of topological 3-cells, (ii) $\mathfrak{A}$ is quasi-homeomorphic to a 3-cell, (iii) $\pi(\mathfrak{A}) \neq 1$, (iv) $C(\mathfrak{A})=1$.

Following Chevalley, the author calls $C(X)$ the "fundamental group" and calls a space X "simply connected" if $C(X)=1$, and uses the terms "fundamental group by paths" and "pathwise simply connected" for the displaced classical concepts. One may infer from the author's introductory remarks that his program is aimed at a moral justification of the violence done to the terminology. Thus he remarks that the example $\mathfrak{A}$ shows that the classical fundamental group $\pi(E)$ does not have the nice behaviour proved for $C(E)$ with respect to quasi-homeomorphism and intersection of sequence. However, good behaviour of an invariant under various operations is not necessarily an argument in its favor; the property of a space to be a space has the best of all possible behaviours and the least value. The example $\mathfrak{A}$ shows that the group $C(E)$ is a blunter tool than the group $\pi(E)$. Furthermore $C(E)$ is not defined for every space—for example, $C(S_1^\infty)$ is not defined [cf. the preceding review]—although every arcwise connected space has a fundamental group. [The author's remark in §1 that local arcwise connectivity is necessary for the definition of the fundamental group is misleading.] *R. H. Fox* (Princeton, N. J.).

Ganea, Tudor. Monotone transformations and the fundamental group. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 305–315 (1950). (Romanian. Russian and French summaries)

The theorems of this paper are all contained in the paper reviewed above. *R. H. Fox* (Princeton, N. J.).

Ganea, Tudor. Existence of simply connected covering spaces. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 317–324 (1950). (Romanian. Russian and French summaries)

If X has a universal covering space and if Y is the image under X of a map φ that is either a local homeomorphism or the projection map of a fibre space (X, φ), then Y also has a universal covering space. An example is given of a space A that has itself for universal covering space although there is a certain point a in A such that $A-a$ has no universal covering space. An example is given of a space E that has a universal covering space although it is not locally simply connected at a certain point 0 [but E is semi-1-connected in the sense of homotopy at 0]. *R. H. Fox.*

Chen, Kuo-Tsai. Isotopy invariants of links. Ann. of Math. (2) 56, 343–353 (1952).

By a link (of n components) is meant the union L of n mutually disjoint simple closed curves $L_1, L_2, \cdots, L_n$ in euclidean 3-space E. By the group $G(L)$ of a link L is meant

the fundamental group $\pi(E-L)$ of the complement of L in E. The lower central series of a group G is denoted $G=G_1\supset G_2\supset\cdots$. The author proves that, given any link L and any positive integer d, there is a group $H^d(L)$ and a positive number δ such that, for every polygonal link L' that is a δ-approximation of L, the quotient group $G(L')/G_d(L')$ is isomorphic to H^d. The group H^d is then shown to be an invariant of link-isotopy, i.e., if, for each $t\in[0,1]$, $L(t)$ is a link, and if $L(t)$ depends continuously on t, then $H^d(L(0))=H^d(L(1))$. The group $H^d(L)$ is of no interest unless $n\geqq 2$ and $d\geqq 3$. The author has shown in another place [Proc. Amer. Math. Soc. 3, 44–55 (1952); these Rev. 13, 721] that $H^3(L)$ is completely characterized by the $n(n-1)/2$ linking numbers $\mathfrak{V}(L_i, L_j)$, $1\leqq i<j\leqq n$, and has obtained some new numerical invariants from the groups H^d, $d\geqq 4$. It is not known whether $H^d(L)$ is isomorphic to $G(L)/G_d(L)$ for every L and every $d\geqq 3$; hence it is not yet known whether $H^d(L)$ depends on the affine structure of E.

R. H. Fox (Princeton, N. J.).

Hu, Sze-Tsen. Homotopy properties of the space of continuous paths. II. The general case with arbitrary boundary sets. Portugaliae Math. 11, 41–50 (1952).

Let X be a topological space, and let A, B be subspaces with non-empty intersection. Then (X, A, B) is a triad in the sense of Blakers and Massey [Proc. Nat. Acad. Sci. U. S. A. 35, 322–328 (1949); these Rev. 11, 47]. Let $[X, A, B]$ be the space of paths in X beginning in A and ending in B (with the compact-open topology). The author identifies the triad homotopy group $\pi_{n+1}(X; A, B)$ with the nth relative homotopy group of a suitably chosen pair of subspaces of $[X, A, B]$; the Blakers-Massey exact sequences then appear as the exact sequences of pairs of subspaces of $[X, A, B]$. In case A and B are both contractible over themselves to a point, a new sequence is obtained, namely,

$$\cdots\to\pi_{n+1}(X; A, B)\to\pi_{n-1}(A\cap B)\xrightarrow{s}\pi_n(X)\to\pi_n(X; A, B)\to\cdots,$$

where s is a generalized Freudenthal suspension. In the last section, the author extends his "realization procedure" [Pacific J. Math. 1, 583–602 (1951); these Rev. 13, 676] to the relative and triad homotopy groups.

P. J. Hilton.

★Pontryagin, L. S. Foundations of combinatorial topology. Graylock Press, Rochester, N. Y., 1952. xii+99 pp.

Translation of "Osnovy kombinatornoĭ topologii" [Gostehizdat, Moscow-Leningrad, 1947; these Rev. 11, 450].

GEOMETRY

★Cundy, H. Martyn, and Rollett, A. P. Mathematical models. Oxford, at the Clarendon Press, 1952. 240 pp. $5.50.

The lack of an inexpensive supply of mathematical models has provoked many teachers into building their own. These include the two authors who are teachers in the sixth form in British schools. Students were encouraged to construct most of the models shown in this book as a means of increasing their interest. The materials used are paper, cardboard, plywood, plastics, wire, string, and sheet metal. The tools used are readily available. Easier construction is enabled by the revelation of the "trade secrets" of many model builders.

The largest part of this fascinating book is devoted to the construction of polyhedral models. These include the Platonic and Archimedean polyhedra, their stellated variations, duals of the Archimedean polyhedra, compound polyhedra and deltahedra. The text is augmented by numerous photographs, diagrams, developed nets and tabulated data. Coxeter's "Regular polytopes" [Methuen, London, 1948; these Rev. 10, 261] and similar works are the source of most of the information.

Plane geometry is represented by dissection models, curve stitching, paper folding, tesselations and methods of constructing curves both graphically and mechanically. Projective geometry is illustrated by string and wire models. The geometry of surfaces, with special emphasis on quadric surfaces, is studied by paper sections, string figures, solid models, and the topology of Möbius strips and the Klein bottle. The described mechanical models include equation-solving machines, harmonographs and two- and three-dimensional linkages. Much of the material has been selected from the 18th Yearbook of the National Council of Teachers of Mathematics [Columbia Univ., New York, 1945].

In the wealth of qualitative and quantitative data, the following few errors were noted. Peaucellier is credited with the invention of the first straight-line linkage in 1864 even though the Sarrus linkage is also shown. The latter was published in 1853. Curves of constant width are improperly called curves of constant diameter. The described surface of constant width has toroidal strips in addition to the mentioned spherical caps. On page 27, the meanings of r, r' and the terms in the equation $SP=PM$ are not given. On page 68, "ecircles" is probably meant to be "excircles", and the undefined notation $m\div n$ is not standard.

American readers will be somewhat puzzled by such Briticisms as the use of the term "trapezium" for "trapezoid" as well as references to British trade names for materials of construction.

Both students and teachers of more advanced mathematics could profit in knowledge and interest by an examination of this book.

M. Goldberg.

Cavallaro, Vincenzo G. Note di generalizzazione sulla geometria del triangolo. Euclides, Madrid 12, 30–34 (1952).

Court, N. A. Isogonal conjugate points for a triangle. Math. Gaz. 36, 167–170 (1952).

Court, N. A. Isogonal points for a tetrahedron. Duke Math. J. 19, 71–74 (1952).

A number of properties are derived of which we cite a typical example. If with the vertices of a tetrahedron as centers spheres are drawn orthogonal to the pedal sphere of a point M, the polar planes of M for those four spheres coincide with the faces of the pedal tetrahedron of the isogonal conjugate of M.

H. A. Lauwerier.

Müller, Alfred. Die Schaubarkeit in der Axonometrie. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. 100, no. 3, 22 pp. (3 plates) (1952).

Goormaghtigh, R. Sur une transformation géométrique. Mathesis 61, 90–93 (1952).

Godeaux, Lucien. Sur quelques transformées rationnelles d'une conique. Mathesis 61, 80–84 (1952).

Srinivasiengar, C. N., and Mukherjee, B. N. Normal linear complexes of a quadric surface. Math. Student 19 (1951), 108–112 (1952).

Neumann, H. Eine Fläche 2. Ordnung, F^2, durch neun Punkte gelegt. Math. Ann. 124, 388–392 (1952).

This paper gives a construction, using 23 planes in all, for the tenth point in which the quadric through nine given points meets a given line through one of these points. The number 23 appears to be considerably smaller than any required in previous solutions of the problem.

J. G. Semple (London).

Łojasiewicz, S. Sur une propriété caractéristique de la spirale logarithmique. Ann. Soc. Polon. Math. 24 (1951), 92–94 (1952).

A plane set of points E is said to have property (s) if it is congruent to its image under every homothetic map $w=\alpha z$ ($\alpha>0$, w and z complex). It is shown that a closed set E has property (s) if and only if there exist complex numbers a, b with $a \in E$ and $R\{b\}\neq 0$, such that the set $E-\{a\}$ is the union of logarithmic spirals $z=a+e^{i\lambda}e^{b\tau}$, $-\infty<\tau<\infty$. (Different λ's give different spirals.) *I. M. Sheffer.*

Burrows, W. H. Some properties of hyperbolic coordinate systems. J. Franklin Inst. 254, 127–141 (1952).

Queiroz, Augusto. The theorem of Pappus. Anais Fac. Ci. Porto 35, 217–228 (1951). (Portuguese)

The purpose of this paper is hard to understand. It contains an extremely long proof for a well-known fact, for which simpler proofs are available. The fact in question is that the axioms of incidence, order, and continuity together with the uniqueness of the fourth harmonic point permit one to establish Pappus' Theorem. *H. Busemann.*

Lagrange, René. Sur les produits d'inversions. Acta Math. 82, 1–70 (1950).

Let P denote a variable point in euclidean N-space. We associate to each point A the function $A=A(P)=-\frac{1}{2}\overrightarrow{AP}^2$. If A is the center of the [hyper-] sphere U of radius R, then U is described by the function $U=U(P)=(\overrightarrow{AP}^2-R^2)/2R$. Put $A(B)=AB$ and $U(B)=UB$ [if U degenerates into a hyperplane, $U=U(P)$ will be the algebraic distance of P from U].

The sphere U determines the inversion $\bar{U}$. $\bar{U}$ is real if A and R^2 are real. $\bar{U}$ maps the point M [the sphere Φ] on the point $\bar{U}M$ [the sphere $\bar{U}\Phi$]. According to the author $\bar{U}M=M-2(UM)U$ and $\bar{U}\Phi=\Phi-2(U\Phi)U$; here

$$U\Phi=\cos\sphericalangle(U,\Phi).$$

Define (1) $\overline{U_nU_{n-1}\cdots U_1}$ by

$$\overline{U_n\cdots U_1}M=\bar{U}_n(\bar{U}_{n-1}(\cdots(\bar{U},M)\cdots)).$$

Put (2) $V_i=\overline{U_n\cdots U_{i+2}U_{i+1}}\,U_i$. Then

$$(3)\quad U_i=V_i+2\sum_{i+1}^{n}(U_iU_j)V_j \quad\text{and}\quad V_i=U_i+2\sum_{i+1}^{n}(V_iV_j)U_j.$$

Furthermore, $\overline{U_n\cdots U_1}=\overline{V_1\cdots V_n}$ and

$$(4)\qquad \overline{U_n\cdots U_1}\,M=M-2\sum_1^n(U_iM)V_i.$$

Among the necessary and sufficient conditions for (1) to be a similarity, the following may be mentioned: 1) Let A_i' [R_i'] denote the center [radius] of V_i. Then $\sum_1^n U_i/R_i'$ either vanishes or it represents the infinite hyperplane. 2) Let the $\bar{U}_i$ be real. Then $\sum_1^n 1/R_iR_i'=0$. The cases $n\leqq 5$ are dealt with at length.

Let (1) be a similarity. Then it is homothetic if and only if $\sum_2^{n-1}[\overrightarrow{A_1A_i}\cdot\overrightarrow{A_1M}/R_iR_i']\cdot\overrightarrow{A_nA_i'}$ is equal either to 0 or to $\overrightarrow{A_1M}$ for every M. This condition and the special case that (1) is the identity are studied in detail. The equation $\sum_1^n\overrightarrow{A_iA_i'}/R_iR_i'=0$ holds if and only if (1) is the product of an inversion $\bar{U}$ by a similarity which keeps the center of U fixed. Conditions are established and discussed for (1) to be an inversion.

P. Scherk (Los Angeles, Calif.).

Lagrange, René. Sur les produits d'inversions. Bull. Sci. Math. (2) 74, 79–112 (1950).

[Cf. the preceding review.] Two systems of spheres (5) $U_1,\cdots,U_n$ and (6) $X_1,\cdots,X_n$ are called equivalent if (7) $\overline{U_n\cdots U_1}=\overline{X_n\cdots X_1}$. Suppose the spheres (5) are linearly independent. Then (7) implies that the spheres (6) are so too and that they are linear combinations of the spheres (5). Put $Y_i=\overline{X_n\cdots X_{i+1}}\cdot X_i$ and form the matrices

$$u=\begin{pmatrix}U_1\\U_2\\\cdot\\\cdot\\\cdot\\U_n\end{pmatrix},\quad v=\begin{pmatrix}V_1\\V_2\\\cdot\\\cdot\\\cdot\\V_n\end{pmatrix},\quad\text{and}\quad T=\begin{pmatrix}1 & 2U_1U_2 & \cdots & 2U_1U_n\\ 0 & 1 & \cdots & 2U_2U_n\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & \cdots & 1\end{pmatrix}$$

[cf. (2)]. Thus $u=Tv$ [cf. (3)]. Let $x=\theta y$ be the corresponding equation connecting the X_i and Y_j. Then (7) holds if and only if $x=Au$ where (8) $ATA'=\theta$. The cases $A=T'T^{-1}$ and $A=TT'^{-1}$ yield two systems (6) with $\theta=T$. Suppose the $\bar{U}_i$ are real. Then any solution A of (8) can be expressed as a product of not more than $(n-1)^2$ matrices of the form

$$\begin{pmatrix}1 &&&&&&& \\ & \ddots &&&&& 0 & \\ && 1 &&&&& \\ &&& \alpha & \beta &&& \\ &&& -\beta & \gamma &&& \\ &&&&& 1 && \\ & 0 &&&&& \ddots & \\ &&&&&&& 1\end{pmatrix}.$$

If X_n is a linear combination of the spheres (5) and if $\bar{X}_n$ is real, then (7) has a solution $X_1,\cdots,X_{n-1}$.

Suppose the spheres $Y_1,\cdots,Y_n$ are linearly independent and the spheres (6) are arbitrary linearly independent linear combinations of them, say $x=Ky$. Then $M\to M-2\sum_1^n(X_iM)Y_i$ is a product of inversions if and only if the same holds true of $M\to M-2\sum_1^n(Y_iM)X_i$, and both conditions are equivalent to $K+K'=(2X_iX_j)$ [cf. (3) and (4)]. Let U be a linear combination of (5). Then $\overline{U_n\cdots U_1U}$ and $\overline{UU_n\cdots U_1}$ are expressed as products of $n-1$ inversions.

P. Scherk (Los Angeles, Calif.).

Lagrange, René. Sur l'équivalence anallagmatique. Bull. Sci. Math. (2) 75, 47–64 (1951).

[Cf. the two preceding reviews.] The spheres (5) and $X_1, \cdots, X_p$ are inversely equivalent if

$$U_n \cdots U_1 = -X_p \cdots X_1. \tag{9}$$

Put $Y_i = \overline{X_p \cdots X_{i+1}} \cdot X_i$. Then (9) holds if and only if $\Phi = \sum_1^n (U_i\Phi) V_i + \sum_1^p (X_i\Phi) Y_i$ for every sphere Φ. If the spheres (5) are linearly independent and if the X_i are required to be linear combinations of them, then (9) is solvable if and only if $n = N+2$, i.e., if every sphere is a linear combination of the U_i. In this case, $T+T'$ is regular and $\min p = \text{rank}(T-T')$. The cases $\min p = n$, 0, or 2 are dealt with and some observations are added on the case $2 < \min p < n$. Let $n < N+2$ and let $T+T'$ be regular. Then the spheres (5) are linearly independent, (9) is solvable, and $|\min p - \text{rank}(T-T')| \leqq N+2-n$. *P. Scherk.*

Hoffman, A. J. Cyclic affine planes. Canadian J. Math. 4, 295–301 (1952).

An affine plane π is said to be cyclic if it admits a cyclic collineation group, generated by τ, fixing a point X and transitive on the remaining points of π. Such a plane must be finite since, if X, P, and Q are collinear points and $\tau^d P = Q$, then τ^d fixes XPQ and so the orbit of P lies on a finite number of lines. If π contains n^2 points, the n^2-1 points besides X may be represented by residues modulo $N = n^2-1$, and τ as the mapping $i \to i+1 \pmod N$. The ith line containing X ($i = 0, \cdots, n$) consists of X and all $r \equiv 1 \pmod{n+1}$. The jth line not through X ($j = 0, \cdots, N-1$) consists of $d_1+j, \cdots, d_n+j \pmod N$ where $d_1, d_2, \cdots, d_n$ are an affine difference set modulo N. This means that the differences $d_i - d_j$, $i \neq j$, are the n^2-n residues modulo N incongruent to 0 modulo $n+1$.

Every finite Desarguesian affine plane is cyclic. It is shown conversely that if π admits a collineation moving X, then π is Desarguesian. Results comparable to those for cyclic projective planes are established. If $p|n$, then p is a multiplier of the difference set. We cannot have n divisible by 6, 10, or various other composite numbers. The author notes that various tests exclude all n's not prime powers less than 212. As a final remark it is shown that there can be a cyclic group transitive on all the points of an affine plane only if $n = 2$. *Marshall Hall* (Columbus, Ohio).

Berman, Gerald. Finite projective geometries. Canadian J. Math. 4, 302–313 (1952).

In the projective geometry of dimension t over the field with p^n elements, $PG(t, p^n)$, Singer has shown [Trans. Amer. Math. Soc. 43, 377–385 (1938)] that there is a cyclic collineation group transitive on the points and $t-1$ spaces. Let $E_r^1 \subset E_{r+1}^1 \subset \cdots \subset E_{t-1}^1 \subset E_t$, with $E_t = PG(t, p^n)$ be a particular ascending chain of spaces. For each space E_s there is a cyclic collineation group transitive on its E_{s-1} spaces. Hence by an appropriate combination of the cyclic collineations for the particular ascending chain above, we may map any E_r into E_r^1. Difference sets associated with these collineations enable us to find the points in any E_r. Various properties of these difference sets are established. *Marshall Hall* (Columbus, Ohio).

Cuesta, N. Projective structures. Revista Mat. Hisp.-Amer. (4) 12, 107–128 (1952). (Spanish)

The author considers complex and real n-dimensional projective spaces as vector spaces, with the natural lattice-order. He studies the relation between such a space A and the subspace A' of A consisting of the real vectors in A; in particular, there are theorems on the relation between the dimensions a and a' of A and A'. When $n=3$, the possible combinations (a, a') correspond to real points, imaginary points, real lines, imaginary lines of two types (2, 1) and (2, 0), and so on. A possible extension of the method to $n=4$ is indicated. At the end are some corrections to the author's paper on a problem equivalent to that of the continuum [same Revista (4) 11, 240–242 (1951); these Rev. 13, 633]. *P. M. Whitman* (Silver Spring, Md.).

***Hjelmslev, Johannes. Grundlag for den projektive geometri. [Foundations for projective geometry.]** Gyldendalske Boghandel, Copenhagen, 1943. 164 pp. (1 plate)

This is not a textbook, but rather a survey with many new slants and ideas. The first chapter uses congruence axioms and covers the following subjects: Euclid's axioms, Hilbert's axioms with an interesting reduction of the additivity of segments, the rôle of the parallel axiom, motions, projectivities on a conic based on cross ratio, inversion geometry, a very elegant and novel derivation of the main formulas of hyperbolic plane geometry, spatial hyperbolic geometry, spherical geometry.

The second chapter deals with the foundations of projective geometry proper. It begins with an historical introduction in which—as elsewhere in the book—the rôle of von Staudt as the creator of pure projective geometry (in contrast to the earlier work of Desargues, Poncelet, etc., where projective results are obtained by non-projective methods) is stressed. Then the following three axioms are introduced. (1) A straight line is a point set of at least three points, and not the whole space. Two points determine a line. (2) A line which intersects two sides of a triangle (= three lines through three non-collinear points) at distinct points intersects the third. The set of all lines containing two distinct points of a set is called the linear extension of the set. A plane is the linear extension of a point and a line not containing it. (3) Not all points lie in a plane. A space is the linear extension of a point and a plane not containing it. The geometry of a space is developed, in particular, eight statements equivalent to Pappus' Theorem are given, one of which is, of course, the Fundamental Theorem of Projective Geometry; an example of the others is: If four lines intersect three mutually skew lines, then any line intersecting three of the first four lines also intersects the fourth. Then geometric algebra is derived and properties of real geometry (where -1 is not a square), quadratic geometry (where each positive number is a square), and archimedean (here called eudoxian) geometry are discussed.

The third and last chapter deals with projective geometry based on the above three axioms and Pappus' Theorem. The more unusual points covered are: the intersection properties of a line and a conic in real quadratic geometry, von Staudt's theory of the complex plane, Seydewitz' generation of quadrics as the locus of intersection of corresponding elements in bundles related by a correlation, space curves of order 3, cubic geometry (where the algebra is such that two real conics intersecting once intersect at least twice).

The book is most enjoyable and illuminating for the expert. Proofs are omitted when standard and sketched when new. Since the nature of a survey implies a multitude of definitions, the book would greatly profit from an index. *H. Busemann* (Los Angeles, Calif.).

★Semple, J. G., and Kneebone, G. T. Algebraic projective geometry. Oxford, at the Clarendon Press, 1952. vii+404 pp. $7.00.

Trotz konsequenter analytischer Behandlung verliert das Buch nichts von seinem Charakter als Lehrbuch der Geometrie. "Nothing, in our opinion, could be more undesirable than that this traditionally elegant subject should be allowed to take on the appearance of being merely a dressing-room in which algebra is decked out in geometrical phraseology." (Aus der Vorrede.) Kenntnis der linearen Algebra und des Matrizenkalkuls wird vorausgesetzt, sonst ist der Aufbau des Buches in sich geschlossen. Der Stoff ist in 16 Kapitel aufgeteilt, jedes davon in mehrere kurze überschriftete Paragraphen. Übungsaufgaben sind im Text eingestreut und ausserdem am Ende eines jeden Kapitels in grösserer Anzahl zusammengestellt. Die Gliederung ist straff, die Darstellung prägnant und von lückenloser Strenge; das Buch ist ohne Ermüdung zu lesen und zum Selbststudium sehr geeignet. Die Stoffauswahl beschränkt sich nicht auf die klassische Theorie der linearen und quadratischen Gebilde und ihrer projektiven Verwandtschaften in Ebene und Raum, die in grosser Ausführlichkeit behandelt werden. Sie umfasst noch Raumkurven 3. Ordnung und Flächen 3. Ordnung, höhere algebraische, insbesondere quadratische Korrespondenzen, Invariantentheorie, Liniengeometrie und eine kurze Einführung in die Geometrie des n-dimensionalen Raumes, die seit C. Segre auch zum Verständnis von Figuren des 2- und 3-dimensionalen Raumes bedeutungsvoll geworden ist, wobei insbesondere die Dimensionen 4 und 5 näher studiert werden. Die einführende Behandlung der genannten Gebiete, die sich als natürliche Verallgemeinerungen und Vertiefung der Theorie der Gebilde 1. und 2. Grades in Ebene und Raum darbieten, erfolgt so, dass der Leser mit den allgemeinen Problemen dieser Disziplinen vertraut wird; ihre Einbeziehung in den vorliegenden Raumen beschränkt sich auf die sorgfältige Behandlung einfacher aber typischer Beispiele. Die beiden ersten Kapitel haben einführenden Charakter. Nach einer historischen Übersicht über die Entwicklung geometrischer Begriffe bis zum Erlanger Programm von Felix Klein werden die analytischen Methoden der projektiven Geometrie auseinandergesetzt; projektive Koordinaten, Dualitätsprinzip, projektive Transformationen, der Fundamentalsatz, einiges über Kegelschnitte und die imaginären Kreispunkte begegnen dem Leser schon hier. Die nachfolgenden Paragraphen bringen den systematischen Aufbau der Theorie in den Dimensionen 1, 2, 3. Zum vertieften Verständnis wird sogleich beim Fortschreiten des Aufbaus die affine und metrische Spezialisierung der projektiven Auffassung behandelt; der Leser wird überdies mit der Einordnung der nichteuklidischen Geometrie in die projektive Geometrie vertraut gemacht. Inhalt: The concept of geometry. The analytical treatment of geometry. Projective geometry of one dimension. Projective geometry of two dimensions. Conic loci and conic envelopes. Further properties of conics. Linear systems of conics. Higher correspondences, Apolarity, and the theory of invariants. Transformations of the plane. Projective geometry of three dimensions. The quadric. The twisted cubic curve and cubic surfaces. Linear systems of quadrics. Linear transformations of space. Line geometry. Projective geometry of n dimensions. Appendix: Two basic algebraic theorems.

R. Moufang (Frankfurt am Main).

Szász, Paul. Verwendung einer klassischen Konfiguration Johann Bolyai's bei der Herleitung der hyperbolischen Trigonometrie in der Ebene. Acta Sci. Math. Szeged 14, 174–178 (1952).

In dieser Note wird die bekannte Liebmannsche Herleitung der hyperbolische Trigonometrie in der Ebene vereinfacht indem die klassische Konfiguration von J. Bolyai zur Bestimmung des Parallelwinkels als Funktion des Lotes und einen Kunstgriff von M. Réthy verwendet wird.

H. A. Lauwerier (Amsterdam).

Ellis, David. Correction to "Notes on abstract distance geometry, II." Monatsh. Math. 56, 180 (1952).

Cf. Monatsh. Math. 55, 185–187 (1951); these Rev. 13, 377.

Convex Domains, Extremal Problems, Integral Geometry

★Yaglom, I. M., i Boltyanskiĭ, V. G. Vypuklye figury. [Convex figures.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951. 343 pp. 6.60 rubles.

The book consists of two parts, the first, pp. 13–125, discusses the background of, and poses, 116 problems; the second, pp. 139–343, gives solutions to the problems; pp. 126–138 contain appendices to the first part on Blaschke's Selection Theorem for (plane) curves and figures and on the concept of curve. There are 310 very good figures.

The problems are divided into eight sections. The first deals with general properties of convex figures (always in the plane). The second centers around Helley's Theorem. The third first gives an intuitive introduction to continuity and then lists problems which may be treated with continuity arguments. For instance, given two disjoint convex figures, there is a line bisecting the area of both, or a hexagon with equal angles can be circumscribed about a given convex figure. The fourth section deals with the addition of convex figures and curves, the fifth with isoperimetric problems, the sixth with other maximum and minimum problems, the seventh with curves of constant width, the eighth with figures which may be turned completely in an equilateral triangle, always touching the sides, just as a curve of constant width may be turned in a square.

The directions for the use of the book, pp. 10–12, are most instructive. One learns that the book is intended for high school students of the last two years and first year university students. The high school teacher is expected to master the book with ease. It is true that the problems require no higher mathematics, but they presuppose a thorough mastership in plane geometry and operating with inequalities, and the ability of working with concepts like convexity which are not accessible to algorithms. If the book really answers its purpose in Russia, high school teaching must be greatly superior to that in many other countries.

H. Busemann (Los Angeles, Calif.).

Goldberg, Michael. The squaring of developable surfaces. Scripta Math. 18, 17–24 (1952).

Dissections into unequal squares of some regions on cylinders, Möbius strips, and cones are exhibited.

W. T. Tutte (Toronto, Ont.).

***Aleksandrov, A. D. A theorem on triangles in a metric space and some of its applications.** Trudy Mat. Inst. Steklov., v. 38, pp. 5–23. Izdat. Akad. Nauk SSSR, Moscow, 1951. (Russian) 20 rubles.

A segment $T(a, b)$ from a to b in a metric space is a curve from a to b whose length equals the distance ab of a and b. Let $a \neq x \in T(a, b)$, $a \neq y \in T(a, b)$. On a surface with constant curvature K construct a triangle $a'x'y'$ such that $a'x' = ax$, $a'y' = ay$, $x'y' = xy$ (if $K > 0$, this is possible only when these three numbers are sufficiently small) and denote by $\gamma_K(x, y)$ the radian measure of the angle at a' in $a'x'y'$. The upper angle α between $T(a, b)$ and $T(a, c)$ is defined as $\alpha = \lim_{\epsilon \to 0} \sup_{y \to a} \gamma_K(x, y)$. This number is independent of K. If the limit exists, we call it the angle between $T(a, b)$ and $T(a, c)$. The K-excess of a triangle abc formed by segments $T(a, b)$, $T(b, c)$, $T(c, a)$ is the sum of the upper angles minus the sum of the angles in a triangle $a'b'c'$ on a surface of curvature K with $ab = a'b'$, $bc = b'c'$, $ca = c'a'$. The main result is: If α_K is the angle at a' in the triangle $a'b'c'$ and v is the least upper bound of the K-excesses of the triangle axy, where $a \neq x \in T(a, b)$, $b \neq y \in T(a, c)$, then $\alpha - \alpha_K \leqq v$.

The applications of this theorem concern spaces in which the curvature is less than or equal to K, that is, the K-excess is non-positive. The function $\gamma_K(x, y)$ is then non-decreasing: if x_0 is the center of a and x on $T(a, b)$, y_0 the center of a and y on $T(a, c)$, then $\gamma_K(x_0, y_0) \leqq \gamma_K(x, y)$. It follows from this inequality that $x_0 y_0 \leqq x_0' y_0'$ (but not conversely). The reviewer has shown how strong the implications of the last inequality above are for $K = 0$, i.e., of $2x_0 y_0 \leqq xy$, [Acta Math. **80**, 259–310 (1948); these Rev. **10**, 623]. The angle between any two segments $T(a, b)$ and $T(a, c)$ exists, even in a stronger sense than above. Segments are locally unique.

H. Busemann (Los Angeles, Calif.).

Hadwiger, H. Hillsche Hypertetraeder. Gaz. Mat., Lisboa **12**, no. 50, 47–48 (1951).

Let $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$ be unit vectors in E^n such that the scalar product of $\mathfrak{a}_i$ and $\mathfrak{a}_j$ equals w, where $-(n-1)^{-1} < w < 1$, for all $i \neq j$. The simplex traversed by the endpoints of the vectors of the form $\sum_{\nu=1}^{n} \lambda_\nu \mathfrak{a}_\nu$, $1 \geqq \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq 0$ issuing from the origin is "zerlegungsgleich" to a cube. For $n = 3$ the theorem is contained in a result of Jessen [Mat. Tidsskr. B. **1941**, 59–65 (1941); these Rev. **7**, 68]. *H. Busemann.*

Hadwiger, H. Ueber zwei quadratische Distanzintegrale für Eikörper. Arch. Math. **3**, 142–144 (1952).

Let K be a convex body of the k-dimensional Euclidean space with the volume V and surface F. Then the following inequalities are valid:

$$\int \frac{dV dV_0}{r^2} \leqq \frac{F^2}{2(k-1)(k-2)}, \quad k > 2,$$

$$\int r^2 dF dF_0 \geqq 2k^2 V^2, \quad k > 1,$$

where dV, dV_0 and dF, dF_0 are two independent elements of volume and surface of K, and r denotes their distance. Proofs are based on results of integral geometry.

S. Chern (Chicago, Ill.).

Hadwiger, H. Einfache Herleitung der isoperimetrischen Ungleichung für abgeschlossene Punktmengen. Math. Ann. **124**, 158–160 (1952).

This proof of the isoperimetric inequality for closed bounded sets in euclidean n-space is a simplification of an earlier proof by the author [Portugaliae Math. **8**, 89–93 (1949); these Rev. **12**, 353]. It is similar to one given by Dinghas [Math. Nachr. **2**, 107–113 (1949); these Rev. **11**, 386]. The crucial step in both papers is the proof of the following lemma: To every $k > 0$ there is a $\mathfrak{K}^*$ such that $\mathfrak{S} \subset (\mathfrak{K}^*)_k$. [The notations are those used in the review of Dinghas' paper.] In Dinghas obtained it as a corollary of the indirectly proven formula $\sup_{\mathfrak{K}^*} V(\mathfrak{K}^* \mathfrak{S}) = V(\mathfrak{S})$. Hadwiger's proof of the above lemma is direct and even provides an explicit estimate of the number of Steiner symmetrizations needed in the construction of $\mathfrak{K}^*$. *P. Scherk.*

Kearsley, M. J. Curves of constant diameter. Math. Gaz. **36**, 176–179 (1952).

Moór, Arthur. Über die Scheitelpunkte der zwei- und dreidimensionalen Kurven. Monatsh. Math. **56**, 150–163 (1952).

In a previous paper [Duke Math. J. **18**, 509–516 (1951); these Rev. **12**, 856] the author has extended the four-vertex theorem to a class of space curves. In this discussion the term vertex is used to indicate an extreme value of a suitably chosen function of curvature and torsion. In the present paper the same method of argument is used to establish results concerning the vertices (in the sense above) of a space curve that meets a sphere only in vertices of the curve. The result obtained is that on such a curve there must exist at least two vertices not on the sphere. The requirements on the curves are here so strong that it seems to the reviewer an existence proof would be desirable to show the class is non-vacuous. The same method of proof is continued to establish an analogous four-vertex theorem for curves in Euclidean n-space through each two points of which passes a hyperplane not otherwise meeting the curve.

S. B. Jackson (College Park, Md.).

Müller, Hans Robert. Über Integrale bei mehrgliedrigen Bewegungsvorgängen. Math. Nachr. **7**, 159–164 (1952).

Erweiterung einiger Formeln der Blaschkeschen Integralgeometrie. Im beweglichen System werden einfache (aus Punkten, Geraden, und Ebenen aufgebaute) Figuren befestigt, die keine Bewegungsinvarianten besitzen. Für geschlossene Bewegungsvorgänge von zwei und mehr Freiheitsgraden werden Ausdrücke für die Dichten dieser Figuren aufgestellt und Integrale berechnet über ein Gebiet des Phasenraumes, das von einer geschlossenen orientierbaren Hyperfläche berandet ist. Nach der Stokeschen Formel werden diese in Randintegrale verwandelt.

O. Bottema (Delft).

Algebraic Geometry

Gorenstein, Daniel. An arithmetic theory of adjoint plane curves. Trans. Amer. Math. Soc. **72**, 414–436 (1952).

Let Γ be an irreducible plane curve of degree m, defined over an arbitrary ground field k, with function field K. A point of Γ is a complete set of conjugate points in the ordinary sense. Let $\delta(\mathrm{B})$ denote the number of conditions imposed by the divisor B on a linear series of sufficiently high order on Γ. Let P be a point of Γ and $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ the prime divisors of K of center P, with $\mathrm{v}_1, \cdots, \mathrm{v}_r$ the corresponding valuations, and let us call the intersections of v_i-ideals $(i = 1, \cdots, r)$ in $\mathfrak{o}_P$ a complete P-ideal in the ring of quo-

tients $\mathfrak{o}_P$ of P. Then there is a one-one correspondence between the integral divisors of the form $\mathbf{A}=\mathfrak{p}_1^{\mu_1}\cdots\mathfrak{p}_r^{\mu_r}$ and the complete P-ideals $\mathfrak{q}$ such that $\mu_i=v_i(\mathfrak{q})$, $i=1, \cdots, r$. For corresponding pairs $\mathbf{A}$, $\mathfrak{q}$ the following equalities are verified: $\delta(\mathbf{A})=\delta(\mathfrak{q})$, $d(\mathbf{A})=d(\mathfrak{q})$, where $d(\mathfrak{q})=\sum_{i=1}^{r} d_i v_i(\mathfrak{q})$, $d_i=[\Delta_i:k]$, Δ_i being the residue field of $\mathbf{v}_i$ and $\delta(\mathfrak{q})=\dim(1)/\mathfrak{q}$. If P is a point of Γ, the conductor $\mathfrak{C}_P$ between $\mathfrak{o}_P$ and its integral closure $\bar{\mathfrak{o}}_P$ is a complete P-ideal. The divisor $\mathbf{C}_P$ corresponding to $\mathfrak{C}_P$ is called an adjoint divisor at P. Then $\mathbf{C}=\prod_{P\varepsilon\Gamma}\mathbf{C}_P$ is a well-defined divisor, called the adjoint divisor of Γ. An adjoint curve to Γ is then defined as every curve which cuts out on Γ a multiple of $\mathbf{C}$. The proof of the Noether Fundamentalsatz, in this more general case, is based on the fact that the $\mathfrak{C}_{P_i}$ are regular ideals, i.e., complete ideals such that, for every complete ideal $\mathfrak{q}$ such that $\mathfrak{q}\subset\mathfrak{C}_{P_i}$, the equation $\delta(\mathfrak{q})-\delta(\mathfrak{C}_{P_i})=d(\mathfrak{q})-d(\mathfrak{C}_{P_i})$ holds. For after it is shown that the adjoint curves of order $m-3$ cut out on Γ outside of the fixed component $\mathbf{C}$ the complete canonical series, it is proved that this proposition is equivalent to (1) $d(\mathbf{C})=2\delta(\mathbf{C})$. This constitutes the fundamental point of the paper. In order to obtain an arithmetic proof of (1), the author generalizes a theorem of Seidenberg [same Trans. 57, 387–425 (1945), Th. 6; these Rev. 7, 2] to the present case in which k is an arbitrary field. Another proof of (1) is given based on the representations of the differentials of the first kind of K in the form $(\Phi(x, y)/F_y'(x, y))dx$, where $\Phi(X, Y)=0$ is an adjoint curve of order $m-3$.

The arithmetic properties established in Part I of the paper are also valid for algebraic number fields. Some minor misprints must be corrected, especially on pages 417 and 432.

P. Abellanas (Madrid).

Masotti Biggiogero, Giuseppina. Caratterizzazione di singolarità della curva hessiana. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 14(83), 444–458 (1950).

Poursuivant l'étude systématique qu'elle a entreprise depuis 1941 [voir références ci-dessous] des singularités que présente la hessienne d'une courbe en un point singulier de celle-ci, l'auteur examine dans la présente note le cas d'une courbe qui possède un point O multiple d'ordre r ($r>2$) origine d'une seule branche superlinéaire d'ordre r avec rebroussement de classe r, d'espèce s et de rang 1, c'est-à-dire, qui admet pour schéma de points infiniment voisins successifs $O^rO_1^r\cdots O^r_{s-1}O_s^1O^1_{s+1}\cdots O^1_{s+r-1}$ dont les $s+1$ premiers se succèdent sur une branche linéaire φ sans inflexion. Cette étude prolonge deux publications antérieures de l'auteur sur ce sujet: l'une [mêmes Rend. (3) 10(79), 89–96 (1946); ces Rev. 10, 398] où est examiné le cas d'une courbe ayant un point double de rebroussement ou tacnodal d'espèce quelconque ($r=2$, s quelconque); l'autre [Ann. Mat. Pura Appl. (4) 30, 277–289 (1949); ces Rev. 12, 735] où est examiné le cas d'une courbe ayant un point multiple origine de r branches toutes linéaires (r quelconque, $s=1$).

Voici le résultat de la présente étude: La hessienne admet en O trois branches superlinéaires d'ordres respectifs r, $r-1$, $r-2$, toutes d'espèce s et de rang 1, et passant chacune par les $s+1$ premiers points de la suite sur φ. Schémas:

$$\begin{array}{llll} O^r & O_1^r & \cdots O^r_{s-1} & O_s^1\cdots O^1_{s+r-1}, \\ O^{r-1} & O_1^{r-1}\cdots & O^{r-1}_{s-1} & O_s^1\cdots O^1_{s+r-2}, \\ O^{r-2} & O_1^{r-2}\cdots & O^{r-2}_{s-1} & O_s^1\cdots O^1_{s+r-3}. \end{array}$$

Pour obtenir ce résultat, l'auteur commence par étudier la courbe:

$$(y+x^2+x^s)^r+2(y+x^2)x^{rs-s+1}+x^{rs+1}=0.$$

Elle examine ensuite, comparativement, les courbes ayant en O un rebroussement d'ordre $r>2$, de classe r et de rang 1, de seconde espèce ($s=2$): ce cas est déduit par transformation quadratique de celui où la classe est $r+1$, cas pour lequel l'auteur donne une représentation analytique locale explicite. Le cas général, $s>2$, est alors démontré par récurrence sur s. Le mémoire se termine par l'examen de l'équivalence plückérienne des singularités étudiées. La présence d'une telle singularité abaisse le nombre des points d'inflexion de $3(r-1)(rs+1)$. *L. Gauthier* (Nancy).

Wylie, C. R., Jr. Line involutions of order three with a quadratic complex of invariant lines. Univ. Nac. Tucumán. Revista A. 8, 31–40 (1951).

In this paper the author establishes that in S_3 there are two and only two distinct cubic line-involutions which possess a quadratic complex of invariant lines, namely, the involutions whose point equivalents in S_5 are either (a) the transformation of V_4^2 into itself effected by the transversals of line and an S_3 in general position in S_5, or (b) the transformation of V_4^2 into itself effected by the transversals of two general planes of S_5. *M. Piazzolla Beloch* (Ferrara).

Manara, Carlo Felice. Sulle curve di diramazione dei piani multipli. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 13(82), 179–184 (1949).

Chisini a montré l'équivalence birationnelle des plans multiples ayant une même courbe de ramification convenablement choisie [mêmes Rend. (3) 8(77), 339–356 (1944); ces Rev. 8, 402]. Dans le présent mémoire, l'auteur démontre, par un procédé indépendant de celui de Chisini, un théorème d'équivalence qui est un cas très particulier de celui de Chisini. Il s'agit de l'équivalence birationnelle des plans multiples ayant comme courbe de ramification une courbe rationnelle donnée dépourvue de point d'inflexion. Une telle courbe est d'ordre pair, soit C^{2k}, de classe $k+1$. La démonstration s'appuie essentiellement, d'une part, sur une étude antérieure de l'auteur, concernant le voisinage des singularités des courbes de ramification [ibid. (3) 9(78), 191–203 (1945); ces Rev. 8, 402], d'autre part, sur le fait que les C^{2k} irréductibles forment une variété connexe, parce qu'il en est de même de leurs duales, les C^{k+1} dépourvues de rebroussement. L'auteur en déduit que les plans multiples ayant C^{2k} pour courbe de ramification sont des plans $(k+1)$-uples, et qu'un des modèles projectifs de cette classe est une réglée, définie en prenant comme fonction algébrique en un point du plan de C^{2k} l'ensemble des valeurs des coefficients angulaires des tangentes à C^{2k} issues de ce point.

L. Gauthier (Nancy).

Godeaux, Lucien. Sur quelques surfaces algébriques représentant des involutions cycliques. I. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 819–825 (1951).

Godeaux, Lucien. Sur quelques surfaces algébriques représentant des involutions cycliques. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 826–835 (1951).

Godeaux, Lucien. Sur quelques surfaces algébriques représentant des involutions cycliques. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 938–949 (1951).

Godeaux, Lucien. Sur quelques surfaces algébriques représentant des involutions cycliques. IV. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 1106–1119 (1951).

Cette succession de quatre mémoires vient en application des méthodes élaborées par l'auteur pour l'étude des involutions cycliques et des surfaces multiples n'ayant qu'un nombre fini de points de diramation [même Bull. (5) 36, 170–179 (1950); 37, 111–120 (1951); ces Rev. 11, 739; 14,

76, dans lequel une bibliographie est indiquée]. Ces exemples éclairent bien la mise en oeuvre de la méthode, et ont en outre l'intérêt d'aboutir à la construction de surfaces algébriques de type nouveau. Il s'agit de l'étude des surfaces:

$$a_1x_1^\nu x_2+a_2x_2^\nu x_3+a_3x_3^\nu x_1+a_4x_4^{\nu+1}=0$$

pour les valeurs de ν telles que la surface soit invariante dans une homographie H de période $p=(\nu-1)^2+\nu$, p premier impair, n'ayant comme points unis que les sommets du repère:

$$H)\qquad x_1'=x_1\quad x_2'=\epsilon x_2\quad x_3'=\epsilon^\alpha x_3\quad x_4'=\epsilon^\beta x_4$$

où $\epsilon^p=1$, ϵ racine primitive, et

$$\alpha=(\nu-1)^2+1$$
$$\beta=\begin{cases}\frac{1}{3}(2\nu^2-3\nu+4) & \text{si}\quad \nu\not\equiv 0 \pmod 3\\ \frac{1}{3}(\nu^2-2\nu+3) & \text{si}\quad \nu\equiv 0 \pmod 3).\end{cases}$$

Le cas $\nu=4$, $p=13$ avait déjà été étudié par l'auteur dans un travail antérieur sur les involutions de genres un [ibid. 25, 308–313 (1939); ces Rev. 2, 14 (non analysé)]. La surface F^5 est alors de genres $p_a=p_g=4$, $p^{(1)}=6$. L'involution I_{13} engendrée par H sur F^5 a pour image une surface Φ^5 régulière de genres $p_a=p_g=P_4=1$. La structure des points unis de I_{13} est étudiée. Φ^5 est birationnellement équivalente à une surface du quatrième ordre ayant 4 points doubles biplanaires. Cet exemple correspond à l'ordre le plus élevé possible pour une involution de genre un.

Dans le cas $\nu=5$, $p=21$ n'est pas premier.

Le cas $\nu=6$, $p=31$ est étudié dans la note I. L'involution I_{31} engendrée par H sur F^7 a pour image une surface Φ ayant trois points de diramation triples triplanaires. La structure des points unis de I_{31} est étudiée. Cette étude, associée à celle des intersections de F^7 avec un faisceau de surfaces cubiques, permet de montrer que Φ est de genres $p_a=p_g=2$, $p^{(1)}=1$, $P_k=k+1$. Le système canonique de cette surface est un faisceau $|\Lambda|$ de courbes elliptiques contenant deux courbes décomposées: l'une est formée de trois courbes rationnelles, l'autre est une courbe rationnelle triple. Les systèmes pluricanoniques sont composés au moyen de $|\Lambda|$.

Le cas $\nu=7$, $p=43$ est étudié dans la note II. L'involution I_{43} engendrée par H sur F^8 a pour image une surface Φ ayant trois points de diramation quadruples, où le cône des tangentes est formé de deux plans et un cône du second ordre. En chacun de ces points Φ possède sur la droite intersection des deux plans, un point double biplanaire ordinaire infiniment voisin du point quadruple. La structure des points unis de I_{43} est étudiée, et ceci permet de montrer que Φ est de genres $p_a=p_g=2$, $p^{(1)}=3$, $P_2=5$. Le système canonique de Φ est constitué par un faisceau de courbes elliptiques auquel est adjointe une composante fixe, rationnelle, mais non exceptionnelle. La construction du système bicanonique permet de donner un modèle projectif normal Φ^8 de S_4, intersection d'une V_3^2 et d'une V_3^4, dont les courbes bicanoniques sont les sections hyperplanes. Le second mémoire se termine par l'étude de ce modèle projectif et de sa projection sur un S_3.

Dans le cas $\nu=8$, $p=57$ n'est pas premier.

Le cas $\nu=9$, $p=73$ est étudié dans la note III. L'involution I_{73} engendrée par H sur F^{10} a pour image une surface Φ ayant trois points de diramation quadruples où le cône des tangentes est formé de deux plans non sécants et d'un cône du second ordre. Ce cône rencontre chaque plan suivant une droite sur laquelle Φ possède un point double biplanaire ordinaire infiniment voisin du point quadruple. La structure des points unis de I_{73} est étudiée, et ceci permet de montrer que Φ est de genres $p_a=p_g=3$, $p^{(1)}=4$, $P_2=7$. Le système canonique de Φ possède trois composantes fixes, rationnelles, de degré virtuel -4; il est complété par les couples de courbes elliptiques d'un faisceau. Le système bicanonique est simple. On a ainsi obtenu un deuxième exemple de surfaces algébriques dont le système canonique possède des composantes fixes rationnelles non exceptionnelles. L'étude du système bicanonique permet d'obtenir un modèle projectif normal Φ^{12} de S_6, intersection d'une V_3^4 et d'une V_3^3, dont les courbes bicanoniques sont les sections hyperplanes.

D'une façon générale, pour $\nu\equiv 2 \pmod 3$, p n'est pas premier. La note IV est consacrée à l'étude des cas où:

$$\nu=3\eta+1\quad p=9\eta^2+3\eta+1,$$

η étant choisi de façon que p soit premier. L'involution I_p engendrée par H sur F a pour image une surface Φ ayant trois points de diramation. Chacun est multiple d'ordre $\eta+2$ pour Φ et le cône des tangentes est formé d'un plan, d'un cône du second ordre non sécant au plan, et d'un cône d'ordre $\eta-1$ sécant à chacun d'eux suivant une génératrice. Φ possède une suite de points doubles infiniment voisins successifs dont le premier est sur la génératrice commune au cône d'ordre $\eta-1$ et au plan. Si η est pair, cette suite comporte $3\eta/2$ points doubles biplanaires dont le dernier est ordinaire. Si η est impair, cette suite comporte $\frac{1}{2}(3\eta-1)$ points doubles biplanaires, sauf le dernier qui est conique. Le système canonique de Φ possède une composante fixe, rationnelle, non exceptionnelle; sa partie variable est formée de $\eta-1$ courbes elliptiques appartenant à un faisceau. Le système bicanonique est irréductible. Les genres de Φ sont $p_a=p_g=\eta$, $p^{(1)}=4\eta-5$, $P_2=5(\eta-1)$.

Il serait intéressant de compléter cette étude par celle du cas où $\nu=3\eta$, $p=9\eta^2-3\eta-1$, η étant choisi de façon que p soit premier, cas dont on connait déjà deux exemples.

L. Gauthier (Nancy).

Godeaux, Lucien. Construction d'une surface dont le système canonique possède des composantes fixes. Rend. Circ. Mat. Palermo (2) 1, 49–56 (1952).

Dans cette note l'A. donne un exemple encore d'une surface algébrique Φ, de genre linéaire $p^{(1)}>1$, dont le système canonique n'est pas irréductible. À cet effet il considère la surface F d'équation:

$$a_1x_1^{3\nu}x_2+a_2x_2^{3\nu}x_3+a_3x_3^{3\nu}x_1+a_4x_4^{3\nu+1}=0.$$

Elle est transformée en soi par l'homographie H d'équations:

$$x_1':x_2':x_3':x_4'=x_1:\epsilon x_2:\epsilon^{9\nu^2-6\nu+2}x_3:\epsilon^{3\nu^2-3\nu+1}x_4,$$

où ϵ est une racine primitive de l'unité d'ordre premier $p=9\nu^2-3\nu+1$. L'homographie H engendre sur F une involution d'ordre p, ayant trois points unis aux sommets O_1, O_2, O_3 du tétraèdre de référence. L'image de cette involution, que l'A. construit suivant les méthodes qu'il a plusieurs fois appliquées, est la surface cherchée Φ. Ses caractères se peuvent déterminer en étudiant la structure des points de diramation O_1', O_2', O_2' correspondant aux points unis O_1, O_2, O_3 et en utilisant la correspondance entre les deux surfaces F, Φ. Les genres p_g et p_a de Φ ont la valeur ν; le genre linéaire $p^{(1)}$ de Φ vaut $3\nu-5$ tandis que le bigenre P_2 est $4\nu-5$. Le système canonique de Φ se compose de trois courbes rationnelles fixes et de $\nu-1$ courbes elliptiques variables dans un faisceau linéaire. Le système bicanonique de Φ est irréductible.

E. G. Togliatti (Gênes).

Godeaux, Lucien. Sur la construction de surfaces irrégulières. Boll. Un. Mat. Ital. (3) 6, 277–280 (1951).

Sur la surface F, image des couples de points d'une courbe C, de genre p, non hyperelliptique, supposée contenir une involution cyclique d'ordre supérieur à 2, dont l'image est une courbe c de genre p', il existe une involution I engendrée par les images des couples se correspondant dans la transformation induite par l'involution de C; soit F' la surface image de cette involution. En partant d'un modèle projectif canonique de C, on peut choisir un modèle de F pour lequel I est déterminée par une homographie cyclique dont on construit les hyperplans unis, d'où possibilité d'étudier la structure des points unis de I sur F. En considérant les correspondances entre f, image des couples de c, F' et F, on détermine le système canonique de F', d'où l'on obtient le genre géométrique de F'. Pour calculer le genre arithmétique, il est nécessaire de faire le calcul complet des points unis. Dans tous les cas étudiés par l'A. l'irrégularité est égale à p'.

B. d'Orgeval (Alger).

Andreotti, Aldo. Sopra le superficie algebriche che posseggono trasformazioni birazionali in sè. Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 9, 255–279 (1950).

An irrational algebraic surface of linear genus $p^{(1)}>1$ can possess at most a finite group of birational transformations into itself. The author obtains an upper bound for the order of this group by observing that, for suitably chosen k, the k-canonical model of the surface is proper, and that on this model, any birational self-transformation is induced by a collineation of the ambient space which leaves invariant the envelope of tangent primes. In particular, if the arithmetic genus of the surface is positive, the author obtains the upper bound

$6q3^{q}c(l+4)$

$$\{I+4+28(p^{(1)}-1)\}^{P_4^2-1},$$

where I and P_4 have their usual meanings. The author next uses transcendental arguments to prove that, for a surface of positive irregularity $q\geqq 2$ and linear genus $p^{(1)}>1$, the period of any birational self-transformation cannot exceed $6q3^{q}(I+4)$. Finally, he proves that every birational transformation of an irregular surface (which is not a Picard surface) into itself gives rise to a birational hermitian self-transformation of the corresponding Picard variety. The case of a Picard surface is exceptional, and an example is given to show that for such a surface the theorem may not hold.

J. A. Todd (Cambridge, England).

Nakano, Shigeo. Note on group varieties. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 55–66 (1952).

It is first noticed that, given two group varieties V and W defined over k, a rational mapping f of V into W, such that $f(xy)=f(x)f(y)$ for two independent generic points x, y of V over k, is everywhere defined on V and is a homomorphism of V into W. Furthermore, the algebraic and set-theoretic images of V by f coincide.

The rest of the paper deals with a timely construction of algebraic homogeneous spaces. Given a group variety G^n and a group subvariety H^r of G defined over k, there exist an algebraic extension K of k, a non-singular variety V^{n-r}, and a canonical mapping f of G onto V, both defined over K; the group G acts as a transitive group of birational and biregular transformations T_s $(s\in G)$ of V, T_s being defined over $K(s)$; for every s, $t\in G$, one has $T_s(f(t))=f(st)$; and the inverse images by f of the points of V are the left cosets of G mod H. The construction of the homogeneous space V is inspired by A. Weil's construction of group varieties [Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948, §V; these Rev. 10, 621]. Substantially V is the variety of the Chow-points of the left cosets of G mod H. Auxiliary results, useful in this construction, include refinements of some intersection theorems of A. Weil, and the fact that the smallest field of definition containing def (A) of a subvariety B of an abstract variety A is obtained by adjoining to def (A) the ratios of the Chow-coordinates of any representative of B. The uniqueness of the homogeneous space V is proved. When H is a normal subgroup of G, V is a group variety and is what the factor group G/H has to be from an algebro-geometric standpoint. A generalization to the case where the subgroup H is only a bunch of varieties is sketched.

P. Samuel (Ithaca, N. Y.).

Igusa, Jun-ichi. On the varieties of the classical groups in the field of arbitrary characteristic. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 67–74 (1952).

Let K be a universal domain of arbitrary characteristic p. The special orthogonal group $SO(n, K)$ and the symplectic group $Sp(n, K)$ are algebraic groups of matrices. The method used for studying them is the following (we explain it for $SO(n, K)$, the other case being similar): if $(a)=(a_{ij})_{1\leq i,j\leq n}\in SO(n, K)$, let $f_q(a)$ be the point $(a_{ij})_{1\leq i\leq q,\, 1\leq j\leq n}$; the algebraic projection $V_q=f_q(SO(n, K))$ is, by E. Schmidt's orthonormalization process, defined by the equations $\sum_{s=1}^{n}a_{is}a_{js}=\delta_{ij}$ for $1\leqq i$, $j\leqq q$. By considering the projection of $SO(n, K)$ on V_1 it is proved that $SO(n, K)$ is connected (for Zariski's topology, and also for the usual one when K is the complex field), from which it follows that it is a variety. The fact that it is a rational variety (parametrized over the prime field) is proved by Cayley's parametrization when $p\neq 2$; when $p=2$, one notices that V_{q+1} is defined by linear equations over $K(V_q)$, which proves the same result by induction on q. The set-theoretic projection of $SO(n, K)$ on V_1 is discussed: it is equal to the algebraic projection V_1 except for $p=2$, n odd; in this case one misses only the point $(1, 1, \cdots, 1)$ of V_1. There are no exceptions in the case of $Sp(n, K)$.

P. Samuel (Ithaca, N. Y.).

Matsusaka, Teruhisa. On a generating curve of an Abelian variety. Nat. Sci. Rep. Ochanomizu Univ. 3, 1–4 (1952).

An abelian variety A is said to be generated by a variety V (and a mapping f of V into A) if A is the group generated by $f(V)$. It is proved that every abelian variety A may be generated by a curve defined over the algebraic closure of def(A). A first lemma shows that, if a variety V is the carrier of an algebraic system $(X(M))_{M\in U}$ of curves ($X(M)$ being defined, non-singular and disjoint from the singular bunch of V for almost all M in the parametrizing variety U) if this system has a simple base point on V, and if a mapping f of V into an abelian variety is constant on some $X(M_0)$, then f is a constant; this is proved by specializing on M_0 a generic point M of U and by using specializations of cycles [Matsusaka, Mem. Coll. Sci. Kyoto Univ. Ser. A. Math. 26, 167–173 (1951); these Rev. 13, 379]. Another lemma notices that, for a normal projective variety V, a suitable linear family of plane sections of V may be taken as a family $(X(M))$. Then the main result follows from the complete reducibility theorem. This result is said to be the basic tool for generalizing Chow's theorem ("the Jacobian variety of a curve defined over k is an abelian projective variety defined over k").

P. Samuel (Ithaca, N. Y.).

Differential Geometry

★**Kagan, V. F. Osnovy teorii poverhnosteĭ v tenzornom izloženii. Čast' pervaya. Apparat issledovaniya. Obščie osnovaniya teorii i vnutrennyaya geometriya poverhnosti. [The fundamentals of the theory of surfaces in tensor presentation. Part one. The apparatus of research. The general foundations of the theory and the intrinsic geometry of a surface.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1947. 512 pp.

★**Kagan, V. F. Osnovy teorii poverhnosteĭ v tenzornom izloženii. Čast' vtoraya. Poverhnosti v prostranstve. Otobraženiya i izgibaniya poverhnosteĭ. Special'nye voprosy. [The fundamentals of the theory of surfaces in tensor presentation. Part two. Surfaces in space. Transformations and deformations of surfaces. Special questions.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1948. 407 pp.

A general characterization of the work may be given by saying that the content is classical but that the material has undergone a thorough reworking into tensor-analytic form. Without giving a complete list of topics we shall say that the book contains a detailed treatment of subjects one expects to find in any extended treatment of differential geometry. The following remarks will help to form a better idea of the book. It begins with a 75 page chapter on groups of transformations (including inversive transformations). Three chapters (120 pages) deal with tensor analysis. About 25 pages are devoted to what the author calls the tensor of Darboux (it may be characterized as giving the deviation of the surface from a quadric, and reports on the work of Verbickiĭ [Trudy Sem. Vektor. Tenzor. Analizu 7, 319–340 (1949); these Rev. 12, 204]. In the chapter on mappings 22 pages are devoted to cartography. An outline of the theory of rectilinear congruences contains 27 pages. Side by side with the usual metric form (called here longometric) another, gonometric, form is introduced which gives the infinitesimal angle between two neighboring curves of a two-parameter family in the same way that the longometric form gives the infinitesimal distance between two points; for the family of geodesics the gonometric form is quadratic only when the surface is of constant curvature; the duality, or near-duality, which obtains in this case is treated in a separate chapter occupying together with an outline of integral geometry 53 pages. The last chapter of 57 pages is devoted to the theory of nets in tensor treatment due mainly to Dubnov. To complete the characterization of the material of the book we may mention some topics not included. Not touched upon are newer developments of the theory of surfaces exemplified, for instance, by another Russian book on the theory of (convex) surfaces that appeared about the same time as the book under review, namely, the book of A. D. Aleksandrov [Intrinsic geometry of convex surfaces, Moscow-Leningrad, 1948; these Rev. 10, 619]. Another development that lies outside the scope of Kagan's book is the generalization to higher dimensions. Differential geometries under groups other than the metric group (and the group of similitudes) are treated only, so to say, as outgrowths of metric geometry and occupy a relatively small space (23 pages out of about 900). Imaginary numbers are avoided. The bibliography contains 187 titles. The exposition is distinguished throughout by a (very successful) effort to achieve complete clarity (at the expense, when it seems necessary, of brevity). The author is also very careful to bring out the historical origins of all concepts he introduces.

G. Y. Rainich (Ann Arbor, Mich.).

Garnier, René. Sur la courbure des surfaces enveloppes en cinématique cayleyenne. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 10, 218–228 (1951).

Dans un travail antérieur [cf. Cours de cinématique, t. II, Gauthier-Villars, Paris, 1949, p. 201; ces Rev. 13, 381], l'auteur a étendu la formule de Savary au mouvement le plus général d'un solide, dans l'espace euclidien. Une surface rigide S dont on connaît le tenseur de courbure, étant entraînée par un solide, dont on connaît l'état cinématique, il s'agit de déterminer le tenseur de courbure de la surface enveloppée S'. Le résultat est le suivant: si l'on remplace les éléments l, r, s, t, $rt-s^2$ de S et de S' par des variables homogènes X, Y, Z, T, U, et X', Y', etc., les derniers s'expriment par une substitution linéaire de X, etc.; les coefficients dépendent de l'état cinématique de S. Maintenant l'auteur démontre que l'on a des résultats analogues dans l'espace cayleyenne. Il ajoute diverses conséquences, par exemple la généralisation d'un théorème de Koenigs sur le cas où S et S' sont constamment osculatrices au cours du mouvement.

O. Bottema (Delft).

Beth, Herman J. E. Sur une classe de systèmes plans à deux paramètres. Bull. Sci. Math. (2) 76, 51–57 (1952).

Given a two-parameter family of motions of a plane Π_m upon a fixed plane Π_f, the possible centers of rotation, for any position of Π_m, lie upon a straight line d. In general the lines d corresponding to the various positions of Π_m form two-parameter families in Π_m and Π_f. However, there exist families of motions such that the lines d form one-parameter families in the two planes. The author discusses the latter families of motions, giving many of the geometrical properties. Evidently, many of these properties are new, but just which ones these are is not easily determinable.

L. A. MacColl (New York, N. Y.).

Marussi, Antonio. Determinazione dell'angolo fra la tangente in un estremo e la corrispondente sezione normale contenente l'altro estremo, per un arco finito di geodetica su di una superficie qualunque. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 566–568 (1952).

Avakumović, Vojislav G. Über geschlossene Kurven auf der Kugel. Srpska Akad. Nauka. Zbornik Radova, Knj. 7, Matematički Inst., Knj. 1, 101–108 (1951). (Serbo-Croatian. German summary)

The author proves the following theorems. 1) A spherical curve K is the tangent representation of a closed spherical curve without double points if and only if K divides the unit sphere into two parts of equal surface. 2) A surface S lies on a sphere or on a plane if and only if the integral of the torsion along any closed curve lying on S is equal to zero. The proofs are based on a direct calculation using vectorial notation. The theorems form a complement to the well-known theorems of Fenchel [Tôhoku Math. J. 39, 95–97 (1934)] and Scherrer [Vierteljschr. Naturforsch. Ges. Zürich 85, 40–46 (1940); these Rev. 3, 89]. *F. Vyčichlo.*

Blanuša, D. Über einige Einbettungsprobleme. Srpska Akad. Nauka. Zbornik Radova, Knj. 7, Matematički Inst., Knj. 1, 91–100 (1951). (Serbo-Croatian. German summary)

Let **S** be a two-dimensional Klein bottle with self-intersection imbedded in a four-dimensional euclidean space.

The author shows that it is possible to extend isometrically the space to a five-dimensional one in such a way that S has no intersection in the extended space. [Cf. C. Tompkins, Bull. Amer. Math. Soc. 47, 508 (1941); these Rev. 2, 301.]
F. Vyčichlo (Prague).

Bilinski, Stanko. Sur un théorème de Jacobi. Srpska Akad. Nauka. Zbornik Radova, Knj. 18, Matematički Inst., Knj. 2, 143–146 (1952). (Serbo-Croatian. French summary)

Sans faire intervenir explicitement le théorème de Gauss-Bonnet, l'auteur démontre le théorème de Jacobi, à savoir que: L'image sphérique de la normale principale d'une courbe fermée régulière divise la surface de la sphère en deux parties égales. *Author's summary.*

Rembs, Eduard. Integralformeln der Verbiegungstheorie. Math. Nachr. 7, 61–64, 387 (1952).

The relation between the integral formula of Herglotz [Abh. Math. Sem. Hansischen Univ. 15, 127–129 (1943); these Rev. 7, 322], from which the congruence of two ovaloids having the same line element follows immediately, and a corresponding formula for the infinitesimal rigidity of an ovaloid [cf. Blaschke, Einführung in die Differentialgeometrie, Springer, Berlin, 1950; these Rev. 13, 274] is derived. The formula of Blaschke is obtained from that of Herglotz by using the development for an analytic isometric deformation up to terms of second order in the deformation components; the formalism of Cartan is employed. *J. J. Stoker* (New York, N. Y.).

Wintner, Aurel. On parallel surfaces. Amer. J. Math. 74, 365–376 (1952).

The main result is the following: If the Gauss curvature of a surface of class C^2 is a positive constant, then the surface is analytic. (A slightly stronger result is contained in statement (2) of the following review.) The idea of the proof is this: the result is proved first under a Hölder condition for the second derivatives. Then it is shown that parallel surfaces to the given surface will satisfy the Hölder condition. The analyticity of the parallel surfaces implies that of the original surface. *H. Busemann* (Los Angeles, Calif.).

Pogorelov, A. V. On a boundary problem for the equation $rt-s^2=\phi(x, y)$ and its geometric applications. Doklady Akad. Nauk SSSR (N.S.) 83, 361–363 (1952). (Russian)

A lemma on the boundary value problem of the title is stated. It is briefly indicated that a proof may be obtained by S. Bernšteĭn's method of a priori estimates. The following theorems on convex surfaces follow from the lemma, where Gauss curvature is defined as limit of the area of the spherical image divided by the area of the surface. (As shown by A. D. Alexandrov and also by Feller and the reviewer, the existence of this limit does not imply the existence of second derivatives.) (1) Let F_1 and F_2 be convex surfaces with the same spherical image w, where w lies on a hemisphere. If F_1 and F_2 have the same positive Gauss curvature at points with parallel normals and the same supporting functions on the boundary of w, then F_1 and F_2 coincide. (2) If the Gauss curvature of a convex surface F as function of the normal is positive and regular (=analytic) (k times differentiable, $k\geqq 3$), then F is regular (at least $k+1$ times differentiable). (3) Let w be a convex domain on the unit sphere, $K(u)$ a positive and continuous function on w, $H(u)$ a continuous function on the boundary of w. Then (because of (1)) a unique convex surface F exists with w as spherical image, $K(u)$ as Gauss curvature, and $H(u)$ as supporting function on the boundary of w. *H. Busemann.*

Mathéev, A. Sur certaines questions de la théorie des courbes et des surfaces réglées de l'espace elliptique. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. 46, 73–115 (1950). (Bulgarian. French summary)

Using the methods of Blaschke [Math. Z. 15, 309–320 (1922)] analogues to the following results on ruled surfaces in euclidean space are obtained: the theorem of Frenet-Serret-Seiliger (=Zeĭliger), Bertrand curves, the formulas of Cesàro. In particular, the lines of striction on non-cylindrical surfaces are studied. In contrast to the euclidean case there are two, which gives rise to some theorems which have no analogue in the euclidean case. An example of such a theorem is: If C is a line of striction and at the same time a geodesic, then the second line of striction is a geodesic if and only if C is a Bertrand curve. Similarly, necessary and sufficient conditions are given for one line of striction to be geodesic and the other asymptotic, and for both to be asymptotic. *H. Busemann* (Los Angeles, Calif.).

Knothe, Herbert. Über isometrische Flächenpaare im elliptischen Raum. Anais Fac. Ci. Porto 35, 65–78 (1951).

A surface in three-dimensional elliptic space is represented in the form $p(u_1, u_2)$ where p is a normed quaternion, i.e., $p\bar{p}=1$. The normal $q(u_1, u_2)$ to $p(u_1, u_2)$ is determined by $q\bar{q}=1$, $q\bar{p}+p\bar{q}=0$, $q\bar{p}_i+p_i\bar{q}=0$ where $p_i=\partial p/\partial u_i$, $i=1, 2$. The vectors $\xi=\bar{p}q$ and $\eta=q\bar{p}$ are called the right and left normals of p. Omitting their (vanishing) scalar parts, ξ and η may be represented as unit vectors in E^3. The following theorem is proved: Consider two surfaces of class C^3 homeomorphic to the sphere which are in addition piecewise analytic. Assume, moreover, that their second fundamental forms are positive definite, and that both the right and the left normals of either surface cover the unit sphere in E^3 simply. If the surfaces are intrinsically isometric, then one can be carried into the other by a motion of the elliptic space. In view of the methods developed in Russia, hypotheses and proof appear strangely antiquated. *H. Busemann.*

Hohenberg, Fritz. Eine Verallgemeinerung der Lilienthalschen Flächenpaare. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. 1951, 129–131 (1951).

Hohenberg, Fritz. Komplexe Erweiterung der gewöhnlichen Schraublinie. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. 1951, 131–132 (1951).

Let w be a complex variable and let

$$\mathfrak{x}(w)=\mathfrak{y}(u, v)+i\mathfrak{z}(u, v)$$

be an analytic curve in complex 3-space. Generalizing the case $(d\mathfrak{x}/dw)^2=0$ of adjoint minimal surfaces, Lilienthal studied the pairs of surfaces $\mathfrak{y}(u, v)$ and $\mathfrak{z}(u, v)$ [J. Reine Angew. Math. 98, 131–147 (1885)]. Generalizing the concept of associate minimal surfaces, the author makes a few observations on the whole two-parametric family of surfaces $\lambda\mathfrak{y}(u, v)+\mu\mathfrak{z}(u, v)$. In corresponding points they have parallel normals and proportional area elements. They are surfaces of translations with conjugate complex generating curves. The second case deals with the special case that the curve $\mathfrak{x}(w)$ is a circular helix $(r\cos w, r\sin w, pw)$ where $r\gtreqless 0$, $p\gtreqless 0$ and $\lambda=\mu=1$. *P. Scherk* (Los Angeles, Calif.).

Mishra, R. S. Skewness of distribution of the generators of a ruled surface. Math. Student 19 (1951), 105–107 (1952).

Krishna, Shri. Congruences formed by the tangents to a surface. Ann. Soc. Sci. Bruxelles. Sér. I. 66, 31–40 (1952).

L'auteur reprend, par le calcul tensoriel, l'étude des congruences rectilignes dont les rayons sont tangents à une surface donnée S (surface focale). Il retrouve un certain nombre de résultats connus (relatifs notamment aux congruences normales), et quelques autres résultats relatifs au cas où la congruence se réduit à l'ensemble des tangentes à une courbe de S. *P. Vincensini* (Marseille).

Lagrange, René. Les courbes dans l'espace anallagmatique. Acta Math. 82, 327–355 (1950).

In euclidean n-space let A denote a point provided with a mass λ and let U_0 be an $[(n-1)$-dimensional$]$ sphere of radius ρ and center 0. We associate with them the point-functions

$$\lambda A=\lambda AP=-\frac{\lambda}{2}\overrightarrow{AP}^2 \quad\text{and}\quad U_0=U_0P=\frac{\overrightarrow{0P}^2-\rho^2}{2\rho}.$$

The anallagmatic group is that subgroup of the conformal group which keeps a given absolute sphere Σ invariant. Choose $\Sigma=U_0$. If $A\not\subset\Sigma$, then the anallagmatic line-element $d\sigma$ at A is the quotient of the euclidean line-element by U_0A.

Let C be a curve outside of Σ and let A move on C. Put $A_0=A/U_0A$. The author constructs a system of $n+2$ mutually perpendicular spheres U_0, $U_1=dA_0/d\sigma$, $U_2, \cdots, U_n$ which is invariant under the anallagmatic group. They satisfy a set of Frenet-formulas $dU_i/d\sigma=-\gamma_{i-1}U_{i-1}+\gamma_iU_{i+1}$ $[i=1, 2, \cdots, n+1;\ \gamma_0=\gamma_{n+1}=0]$. Here $\gamma_1, \gamma_2, \cdots, \gamma_n$ are interrelated by $\sum_0^{n+1}(U_iA_0)^2=0$. The anallagmatic curvatures are certain functions of the γ's and their derivatives.

Let $\lambda=[(d\gamma_1/d\sigma)^2+\gamma_1^2\gamma_2^2]^{-1/4}$. The conformal line-element is $ds=\lambda^{-1}d\sigma$. A second system of mutually orthogonal spheres $U_1, V_2, \cdots, V_n$ is constructed such that each $V_k=\sum_2^{k+1}\theta_k^iU_i$ is a linear combination of $U_2, \cdots, U_{k+1}$. The V_k's touch C at A and satisfy differential equations of the following form: $dV_2/ds=-\lambda\gamma_1\theta_2^2U_1+\Gamma_2V_3$;

$$dV_k/ds=-\lambda\gamma_1\theta_k^2U_1-\Gamma_{k-1}V_{k-1}+\Gamma_kV_{k+1} \qquad [k=3, \cdots, n;\ \Gamma_n=0].$$

The θ_k^i are functions of the $\gamma_1, \cdots, \gamma_k$ and their derivatives with respect to σ. The $(k-1)$th conformal curvature Γ_k is a conformal invariant of order $k+2$. According to the author it satisfies the equations $\Gamma_kV_{k+1}=\sum_2^{k+2}\theta^i_{k+1}U_i$ and $\Gamma_k=[\sum_2^{k+2}(\theta^i_{k+1})^2]^{1/2}$.

While the sphere V_k still depends on the choice of Σ, the pencil of spheres π_k through V_k and the null-sphere A is conformally invariant. By means of the π_k's, Cartan's invariant system of n spheres $\Phi_1, \cdots, \Phi_n$ and two points $\Phi_0=\lambda^{-1}A_0$ and Ψ_0 are constructed: The sphere $\Phi_1=d\Phi_0/ds$ lies in the pencil through U_1 and A. Like U_1, it is normal to C at A. There exists an invariant Γ_1 of order five such that $\Psi_0=d\Phi_1/ds+\Gamma_1\Phi_0$ is an invariant point on Φ_1. Then $\Phi_k=V_k+\lambda\gamma_1\theta_k^2\Phi_0$ will be the sphere in the pencil π_k through that point. Apart from $\Phi_0\Psi_0=-1$, Cartan's $n+2$ spheres are mutually perpendicular. They satisfy the differential equations

$$\frac{d\Psi_0}{ds}=-\Gamma_1\Phi_1+\Phi_2,\quad \frac{d\Phi_2}{ds}=\Gamma_2\Phi_3+\Phi_0,$$

$$\frac{d\Phi_k}{ds}=-\Gamma_{k-1}\Phi_{k-1}+\Gamma_k\Phi_{k+1}$$

$[k=3, 4, \cdots, n;\ \Gamma_n=0]$. The intersection of $\Phi_n, \Phi_{n-1}, \cdots, \Phi_k$ is the osculating $(k-1)$-dimensional sphere of C at A. *P. Scherk* (Los Angeles, Calif.).

McDonald, Janet. Davis's canonical pencils of lines. Pacific J. Math. 2, 209–218 (1952).

Let x denote Fubini's normal coordinates of a generic point P_x of an analytic non-ruled surface S in ordinary space; then S is an integral surface of a system of differential equations of the form $x_{uu}=px+\theta_ux_u+\beta x_v$, $x_{vv}=qx+\gamma x_u+\theta_vx_v$, $\theta=\log\beta\gamma$. A conjugate net N_λ may be defined by the differential equation $dv^2-\lambda^2du^2=0$ and the associate conjugate net by the equation $dv^2+\lambda^2du^2=0$. Let the points x, x_u, x_v, x_{uv} be the vertices of a local coordinate reference frame. A bundle of quadrics (given by equation (2.5) of the paper) exists each quadric of which has contact of at least the third order with both curves of the net N_λ at the point P_x. A pencil of these quadrics (represented by equation (2.6)) consists of quadrics having second order contact with S at P_x. Lines $l_1(k)$ and $l_2(k)$ are determined by the pairs of points $(1, 0, 0, 0)$, $(0, -a_k, -b_k, 1)$ and $(-b_k, 1, 0, 0)$, $(-a_k, 0, 1, 0)$, respectively, in which

$$a_k=\tfrac{1}{2}(\theta_v+\lambda_v/\lambda+k\beta/\lambda^2),\quad b_k=\tfrac{1}{2}(\theta_u-\lambda_u/\lambda+k\gamma\lambda^2).$$

Two lines $l_1(k)$, $l_1(-k)$ (or $l_2(k)$, $l_2(-k)$) are called associate lines. The lines $l_1(k)$ and $l_2(k)$ lie in Davis's first and second canonical pencils [W. M. Davis, dissertation, Chicago, 1935], respectively. The following theorems are proved. 1) The locus of the polar lines of any line $l_2(k)$ with respect to the quadrics of the bundle (2.5) is a cone which is intersected by Davis's canonical plane in the axis of the conjugate net N_λ and in the line $l_1(k-\tfrac{1}{2})$. 2) The polar planes of the points on a line $l_1(k)$ with respect to the quadrics of the bundle (2.5) pass through the point (4.3), which is the point of intersection of the line $\beta x_2/\lambda-\gamma\lambda x_3=0$, $x_4=0$ and the line $l_2(k+\tfrac{1}{2})$. 3) The locus of the points of intersection of the associate conjugate tangents with the line $l_2(k)$ and of the points of intersection of the conjugate tangents with the line $l_2(-k)$ is a cubic curve (5.3). 4) A line $l_2(k)$ intersects the cubic (5.3) in three points, one of which lies on each of the associate conjugate tangents and the third of which lies on the R_λ-correspondent of the tangent t_λ. The two points on the associate conjugate tangents are separated harmonically by the point on the R_λ-correspondent and the point of intersection of $l_2(k)$ by Davis's second canonical tangent.

Finally, a family of quadrics (6.2) generalizing the Davis canonical quadric is introduced by the following characteristic properties. 1) The quadrics have second order contact with the surface S. 2) A line $l_1(k)$ of the first canonical pencil and the line $l_2(-k)$ are reciprocal polars with respect to the quadrics. 3) The quadrics pass through the point P_y which is the harmonic conjugate of the point P_x with respect to the focal points of the line $l_1(k)$. The family (6.2) is found to have the property that Davis's canonical point and canonical plane are pole and polar with respect to any quadric of the family. Two quadrics of the family (6.2) are called associate if they are characterized by values of a parameter k which are numerically equal but opposite in sign. Arbitrary pairs of associate quadrics of the family (6.2) are found to form residual conics of intersection which lie in a common plane. *P. O. Bell* (Berkeley, Calif.).

Saban, Giacomo. Sulle varietà quasi-asintotiche. I. Proprietà elementari collegate alla nozione di specie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 562–568 (1950).

Saban, Giacomo. Sulle varietà quasi-asintotiche. II. Varietà subordinate di varietà quasi-asintotiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **9**, 55–61 (1950).

Saban, Giacomo. Sulle varietà quasi-asintotiche. III. Ancora sulle varietà subordinate. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 113–117 (1951).

Using elementary geometrical considerations, the author obtains results concerning quasi-asymptotic varieties of various kinds, and the relations between quasi-asymptotic varieties and their subordinate varieties. *J. A. Todd.*

Saban, Giacomo. Quasi-asintotiche ad n indici e teoremi di variabilità degli indici. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9**, 429–452 (1950).

This paper appears to be vitiated by an error which was pointed out by M. Villa [see the following review].

J. A. Todd (Cambridge, England).

Villa, Mario. Sulle quasi-asintotiche. Boll. Un. Mat. Ital. (3) **6**, 195–197 (1951).

The author draws attention to certain errors in recent papers by G. Saban [see the four papers reviewed above].

J. A. Todd (Cambridge, England).

Villa, Mario, e Vaona, Guido. Varietà quasi-asintotiche a più indici e curve caratteristiche di una trasformazione puntuale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 470–476 (1950).

Given two varieties V_k, V_m such that V_m lies on V_k, a curve γ of V_m is called quasi-asymptotic $\gamma_{p,q,r}$ of (V_k, V_m) if, at its general point, the join of the pth osculating space of V_k, the qth osculating space of V_m, and the rth osculating space of γ has abnormally small dimension. The authors' main result is that if T is a point-correspondence between two spaces S_r, S_r', represented on the corresponding Segre variety V_{2r} by V_r, then, with some trivial exceptions, the only quasi-asymptotic $\gamma_{1,2,1}$ of (V_{2r}, V_r) are the curves which map the pairs of corresponding characteristic curves of T. In the last section of the note the definition is generalised, just by replacing γ by a V_l $(l<m)$, and then by considering, instead of $V_k \supset V_m \supset V_l$, a chain $V_{k_1} \supset V_{k_2} \supset \cdots \supset V_{k_n}$; V_{k_n} is quasi-asymptotic of indices $\rho_1, \cdots, \rho_n$ of $(V_{k_1}, V_{k_2}, \cdots, V_{k_{n-1}})$ if, at its general point, the ρ_ith osculating spaces of V_{k_i} $(i=1, \cdots, n)$ have a join of abnormally small dimension.

J. A. Todd (Cambridge, England).

Muracchini, Luigi. Ricerche sulle varietà quasi-asintotiche. I. Quasi-asintotiche $\sigma_{1,2}$. Boll. Un. Mat. Ital. (3) **6**, 198–205 (1951).

The author considers varieties V_k which possess ∞^δ $(k\leqq\delta\leqq k+h(k-h))$ calottes $\sigma_h{}^1$ of $\sigma^t{}_{1,2}$, with particular reference to $k=3, 4$. A "calotte $\sigma_h{}^1$" is the geometric object composed of a point of V_k and a tangent S_h at the point; and it is said to be a calotte of $\sigma^t{}_{1,2}$ if it is contained in a calotte $\sigma_h{}^2$ of the second order whose osculating space is joined to the tangent S_k of V_k by a space whose dimension is less by t than the normal dimension of such a join. The following result is obtained finally: The V_3 which possess ∞^1 surfaces which are quasi-asymptotic $\sigma^3{}_{1,2}$ are (a) loci generated by ∞^1 planes, (b) developables with a focal surface containing a pencil of inflexional curves, (c) cones projecting a surface of this type. The V_3 which possess two such systems of surfaces are cones in S_4. *J. A. Todd.*

Muracchini, Luigi. Ricerche sulle varietà quasi-asintotiche. II. Quasi-asintotiche $\sigma_{r,s}$ di specie massima. Boll. Un. Mat. Ital. (3) **6**, 299–304 (1951).

The problems treated are of the same nature as those in the paper reviewed above, but $\sigma^t{}_{1,2}$ is replaced by $\sigma^t{}_{r,s}$ and t is assumed to have its maximum value. Typical result: if every calotte $\sigma_h{}^2$ $(1\leqq h\leqq k-1)$ of a V_k is of $\sigma_{1,3}$ and maximum species, then V_k belongs to an S_N with $N=k+\frac{1}{2}h(h+1)$.

J. A. Todd (Cambridge, England).

Muracchini, Luigi. Sulla deformazione proiettiva delle trasformazioni puntuali. Boll. Un. Mat. Ital. (3) **7**, 29–38 (1952).

In this paper a new correspondence between two point transformations, for projective planes, of the first species of O. Borůvka [Publ. Fac. Sci. Univ. Masaryk **1926**, no. 72; **1927**, no. 85] is introduced and studied. Analogously to the Fubini's projective deformation (or applicability) of surfaces, this correspondence is called a projective deformation (or applicability) of the transformations. Using E. Cartan's method and Borůvka's moving references, a system of Pfaffian equations is obtained for determining projectively deformable transformations. By a neighborhood of the third order of a transformation, a differential form, called the projective linear element of the transformation, is derived to be invariant under a projective deformation.

C. C. Hsiung (Bethlehem, Pa.).

Sorace, Orazio. Sulle condizioni di illimitata integrabilità di un sistema di equazioni differenziali. Boll. Accad. Gioenia Sci. Nat. Catania (4) **6** (1950), 351–356 (1951).

In a Riemannian manifold V_n with connection $\{{}_{\mu\lambda}^{\kappa}\}$ let there be given a second connection $\Gamma_{\mu\lambda}{}^\kappa=\{{}_{\mu\lambda}^{\kappa}\}+A^{\kappa}{}_{\mu\lambda}$. The necessary and sufficient conditions for the existence of n independent vector fields, covariant constant with respect to the latter connection, are expressed in terms of the curvature affinor of V_n and in $A^{\kappa}{}_{\mu\lambda}$ and its covariant derivatives. Some special cases are considered. *A. Nijenhuis.*

Sorace, Orazio. Sopra alcune questioni di calcolo differenziale assoluto. Boll. Accad. Gioenia Sci. Nat. Catania (4) **6** (1950), 357–373 (1951).

The author considers displacements along a certain (not necessarily closed) "parallelogram" in a Riemannian manifold V_n with two metrics. By identifying the two metrics he finds the well-known formula for pseudo-parallel displacements around an infinitesimal quadrangle.

A. Nijenhuis (Princeton, N. J.).

De Sloovere, H. Sur le nombre d'invariants distincts, fonctions de tenseurs, par la méthode de Lie et De Donder. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 437–441 (1952).

As in the previous papers of the author [same Bull. (5) **37**, 583–598 (1951); **38**, 131–135 (1952); these Rev. **13**, 493; **14**, 84], the invariants of complete systems of partial differential equations in n independent variables are discussed. By calculation of invariants, it is shown in the present paper that for $n=2$, one and only one invariant exists. [Reviewer's note: for Riemannian spaces, this result follows from the Frenet formulas and the fundamental theorem of curve theory. That is, the curvature of a curve is its only invariant in two-space.] *N. Coburn.*

Hlavatý, V. Géométrie différentielle de contact. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 157–163. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs; 2450 French francs.

A contact-element of the 1st order (briefly, an element) of a surface is characterized by its fundamental form, with coefficients E, F, G, by its center P, and by its plane τ; a 2nd order holonomic element is characterized by its 1st fundamental form and by its second fundamental form, with coefficients $h_{ij}=h_{ji}$ $(i=1, 2)$ or, what is the same, by a polarity in the plane τ. An element, in general non-holonomic, is represented by the author by a fundamental tensor h_{ij} (symmetric if, and only if, the element is holonomic) and by its Gaussian curvature, defined as the invariant $K=(h_{11}h_{22}-h_{12}h_{21})/\Delta$, where $\Delta=EG-F^2$. We may therefore introduce six homogeneous coordinates, $e_{ij}=-e_{ji}$ $(i=1, 2, 3, 4)$, such that

$$e_{12}:e_{34}:e_{23}:e_{14}:e_{31}:e_{24}=\Delta:-K:h_{22}:h_{11}:-h_{21}:h_{12},$$

and

$$(1) \qquad e\cdot e=2(e_{12}e_{34}+e_{23}e_{14}+e_{31}e_{24})=0.$$

The totality of the elements having the same center P and in the same plane τ may be represented by the set of correlations:

$$(2) \qquad \xi:\eta:\zeta=(e_{14}x+e_{24}y):(e_{12}x+e_{23}y):e_{12}z;$$

$$(2') \qquad x:y:z=(e_{23}\xi+e_{42}\eta):(e_{31}\xi+e_{41}\eta):e_{42}\zeta,$$

where (x, y, z), (ξ, η, ζ) are homogeneous point-coordinates (line-coordinates) in τ. Equations (2′) are the inverse of (2) in consequence of (1), and the correlations (2) or (2′) reduce to a polarity if, and only if, the element e is holonomic. The above totality of elements is a space E_4, and the author's aim is to study this space and its subspaces with the aid of correlation (2). Only results, without proofs, are given in this paper, and the results are stated for the general case, neglecting the particular cases. It follows from (1) that the e_{ij}'s may be considered as line-coordinates in S_3, so that one may naturally speak of "linear complexes", "congruences", "reguli" of elements; in particular, let $\overset{i}{c}$ $(i=1, 2)$ be two "pseudo-elements" (i.e., two points of the 5-space where the "line" e is represented), and consider the congruence of equations:

$$e\cdot\overset{i}{c}=e_{12}\overset{i}{c}_{34}+e_{23}\overset{i}{c}_{14}+e_{31}\overset{i}{c}_{24}+\overset{i}{c}_{12}e_{34}+\overset{i}{c}_{23}e_{14}+\overset{i}{c}_{31}e_{24}=0$$

(and $e\cdot e=0$), of axis f, g; the projective invariant

$$\omega(\overset{1}{c}, \overset{2}{c})=\tfrac{1}{2}\epsilon^{1/2}\ln(f, g, \overset{1}{c}, \overset{2}{c})$$

($\epsilon=\pm1$ according as f, g are real or complex conjugate) may be interpreted as follows with regard to the correlation (2): equations (2), where e is replaced by $f, g, \overset{i}{c}$, respectively, associate to a fixed point P of τ four lines, r_f, r_g, $r_{\underset{c}{1}}$, $r_{\underset{c}{2}}$, which belong to a same pencil, and the non-euclidean angle, of $r_{\underset{c}{1}}$, $r_{\underset{c}{2}}$, with respect to r_f, r_g as absolute, is equal to $\omega(\overset{1}{c}, \overset{2}{c})$. Similarly, by (2′) we get four points on a line, $\underset{1}{C}$, $\underset{2}{C}$, F, G; and the non-euclidean distance $\underset{1}{C}\underset{2}{C}$, with respect to F, G as absolute is also $\omega(\overset{1}{c}, \overset{2}{c})$. It is to be remarked that this duality is not trivial, because the two correlations obtained by replacing in (2) and (2′) e by $\overset{1}{c}$ (e.g.) are not inverse, since $\overset{1}{c}$ does not fulfill (1). In the following paragraph the author deals with curves of E_4, i.e., the sets E_1 of elements depending on a parameter s: $e=e(s)$, and different properties are established with regard to the correlation (2); for E_1 one may define a projective arc, and three pseudo-elements $\overset{i}{n}$ $(i=3, 4, 5)$, used to derive "Frenet's formulae", with three curvatures, $\overset{i}{k}$ $(i=1, 2, 3)$. The meaning of these curvatures is found using the invariant ω mentioned above; the condition $\overset{3}{k}=0$ $(\overset{2}{k}=\overset{3}{k}=0)$ is necessary and sufficient for E_1 to belong to a linear complex (to a congruence); and $\overset{i}{k}=0$ is necessary and sufficient for E_1 to be a regulus. Finally, a differential intrinsic equation is derived for the coordinates of the element generating E_1. The last paragraph is devoted to the study of E_4 itself: if its points (elements) are expressed in terms of four independent parameters ξ^λ $(\lambda=1, 2, 3, 4)$, let $e_\lambda=\partial e/\partial\xi^\lambda$; then the expressions $a_{\lambda\mu}=e_\lambda\cdot e_\mu$ turn out to be the components of a symmetric tensor, defined but for a conformal transformation; E_4 becomes therefore a Riemannian conformal space; as its conformal curvature vanishes, E_4 is conformally flat; moreover, if we assume (with no loss of generality) that the tensor $a_{\lambda\mu}$ is flat, the following statement holds: Each mapping of E_4 onto itself, which belongs to the anallagmatic group (with respect to the tensor $a_{\lambda\mu}$) preserves the flatness of E_4; and conversely.

V. Dalla Volta (Rome).

Klingenberg, Wilhelm. Zur affinen Differentialgeometrie. I. Über p-dimensionale Minimalflächen und Sphären im n-dimensionalen Raum. Math. Z. **54**, 65–80 (1951).

Klingenberg, Wilhelm. Zur affinen Differentialgeometrie. II. Über zweidimensionale Flächen im vierdimensionalen Raum. Math. Z. **54**, 184–216 (1951).

La presente ricerca ha come punto di partenza un lavoro di alcuni anni or sono di K. Weise [Math. Z. **43**, 469–480 (1937); **44**, 161–184 (1938)], che si occupava della geometria di una varietà F_p in uno spazio affine A_n; più precisamente, il Weise riusciva, quando p fosse sufficientemente alto rispetto ad n (esattamente per $n\leqq p+\binom{p+1}{2}$), ad associare ad un punto di una F_p in A_n (sotto opportune condizioni di regolarità) uno spazio "pseudonormale affine" (Affinnormalenraum), di dimensione $n-p$, invariante per affinità spaziali e per trasformazioni dei parametri nella F_p. Restringendosi poi al gruppo delle affinità equivalenti (inhaltstreue Affinitäten), si riesce a definire un elemento di p-volume affine. Per $p=n-1$, si ritrovano noti concetti di geometria affine delle ipersuperficie. Il presente lavoro è diviso in 2 parti: la prima, di carattere più analitico, è dedicata a una F_p in A_n, senza precisazioni riguardo ai numeri p, n; la II, più geometrica, si occupa delle F_2 in A_4. Nel n.1 della parte I l'A. riassume sostanzialmente i risultati del Weise, sviluppando anche il simbolismo—piuttosto complicato—necessario per il seguito; più esattamente, si determina lo spazio pseudonormale associato al punto generico di F_p (e che dipende dalla calotta del 3° ordine di centro P), e le equazioni fondamentali (Ableitungsgleichungen), con le relative condizioni di integrabilità. Infine, nel caso di affinità equivalenti, si dà l'espressione dell'elemento di p-volume. Nel n.2, si determina la variazione prima dell'integrale p-dimensionale dell'elemento di volume (Affineoberfläche), e si definiscono le varietà minime affini (Minimalflächen), come quelle per cui tale variazione prima si annulla. Esse sono caratterizzate dal fatto che l'integrale p-dimensionale che determina il volume affine (p-dimen-

sionale Oberflächenintegral) può trasformarsi in un integrale a $(p-1)$ dimensioni. Nel n.3, invece, si determina anzitutto il luogo delle intersezioni di uno spazio pseudonormale affine in un punto P con gli spazi pseudonormali affini nei punti infinitamente vicini, e si definiscono le sfere affini (Sphären), come quelle F_p tali che tutti gli spazi pseudonormali abbiano a commune uno spazio lineare di dimensione $n-p-1$; si dà l'equazione che caratterizza tali sfere, e si dimostra che dalla proprietà locale che tutti gli spazi pseudonormali nell'intorno di un punto abbiano a comune uno spazio a $n-p-1$ dimensioni, segue la stessa proprietà in grande. La p. II, come si è detto, è dedicata alle F_2 in A_4. L'A. si riferisce al caso più generale, nel quale da ogni punto P di F_2 escono due direzioni coniugate distinte e, inoltre suppone queste reali (per quanto osservi che i risultati si estendono facilmente anche al caso complesso); tali direzioni si possono allora assumere come assi $\mathfrak{x}_1$, $\mathfrak{x}_2$ del riferimento locale $(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{n}_1, \mathfrak{n}_2)$, mentre gli assi $\mathfrak{n}_1$, $\mathfrak{n}_2$ individuano lo spazio (piano) pseudonormale affine. Il riferimento può essere caratterizzato geometricamente al modo seguente; lo $S_3\equiv(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{n}_1)$ sega F_2 in una curva con un punto doppio; ebbene il riferimento canonico (invariante) è caratterizzato dal fatto che tale punto doppio ha le tangenti coincidenti entrambe con l'asse x_1; similmente si può caratterizzare lo $S_3(\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{n}_2)$; quanto al piano pseudonormale $(\mathfrak{n}_1, \mathfrak{n}_1)$, esso si ottiene al seguente modo: si considerino le quadriche aventi contatto del 3° ordine con F_2 in P; esse sono ∞^4, ma i loro centri si distribuiscono su un cono quadrico a 3 dimensioni, di vertice P, rispetto a cui il piano tangente $(\mathfrak{x}_1, \mathfrak{x}_2)$ e il piano pseudonormale $(\mathfrak{n}_1, \mathfrak{n}_2)$ sono coniugati. Nei nn. successivi, servendosi anche dei risultati della p. I, l'A. discute vari tipi particolari di F_2: anzitutto le superficie di traslazione, definite alla maniera usuale, poi le quasi-sfere (Semisphären), definite dalla proprietà che tutti i piani pseudonormali o contengono una retta parallela a una retta fissa, ovvero le hanno le giaciture in uno stesso A_3; in particolare si determinano tutte le quasi-sfere di traslazione. Infine si discutono le sfere affini (v. p. I), assegnandone l'equazione differenziali, che vengono discusse a fondo, considerando vari casi; di questi uno resta dubbio, mentre gli altri o dànno luogo ad assurdi, o conducono a superficie effettivamente esistenti, di cui si dànno anche costruzioni geometriche. Nell'ultimo n. poi, si determinano le superficie minime (p. I) di traslazione. *V. Dalla Volta* (Roma).

Klingenberg, Wilhelm. Über das Einspannungsproblem in der projektiven und affinen Differentialgeometrie. Math. Z. 55, 321–345 (1952).

Il problema che l'A. si propone è il seguente: Data in P_n (spazio proiettivo n-dimensionale) una varietà F_p (con le necessarie condizioni di derivabilità), e detto S_{i+1} lo $(i+1)$-esimo spazio osculatore a F_p in un punto P $(i=1, 2, \cdots, m;$ $S_{m+1}=P_n)$, si definiscono gli spazi pseudonormali N_i con le seguenti condizioni: $S_i\cap N_i=0$; $S_i\cup N_i=S_{i+1}$; in particolare, N_0 è uno spazio a $p-1$ dimensioni nello spazio tangente a F_p in P; (si noti che l'indice i di S_i non indica affatto la dimensione di tale spazio osculatore). Con tale definizione, gli spazi pseudonormali (per ogni punto) non sono affatto definiti in modo univoco; e si cerca di determinarli in modo proiettivamente invariante e indipendente dalla scelta dei parametri sulla varietà. L'A. determina anzitutto una serie di equazioni fondamentali (Ableitungsgleichungen), involgenti le derivate prime e seconde (rispetto ai parametri) delle coordinate del punto variabile su F_p, e le derivate prime delle coordinate dei punti che individuano i vari spazi pseudonormali (la cui dimensione dipende ovviamente da quella dei successivi spazi osculatori). Delle formule da lui ottenute, l'A. si serve per dimostrare che—almeno in generale—è possibile determinare in modo univoco, mediante certe condizioni di "apolarità generalizzata", tutti gli spazi pseudonormali, a partire da N_1 in poi, non appena si sia fissato in modo intrinseco N_0. Questo è ovviamente possibile nel caso affine, in cui basta prendere lo spazio improprio dello spazio tangente a F_p; e in tal caso si ritrovano i risultati di Weise usati dall'A. in un altro lavoro [v. recensione precedente]. Seguono poi considerazioni generali, senza risultati specifici, sulla possibilità di determinare N_0, e l'esame di alcuni casi particolari, in cui si confrontano i presenti risultati con altri di diversi AA. Il lavoro è di carattere analitico, e non si dà cenno di una possibile costruzione geometrica dei vari spazi pseudonormali.

V. Dalla Volta (Roma).

Grove, V. G. On a hypersurface imbedded in a space of n dimensions. Univ. Nac. Tucumán. Revista A. 7, 243–257 (1950). (Spanish)

The author presents here a generalization to more dimensions of some ideas of the geometry of a surface imbedded in a projective 3-space. The main features of this paper are the following: if $n-1$ is the dimension of a hypersurface of a projective n-space, let l be a subspace of dimension $n-2$ of the tangent hyperplane to the hypersurface at a point P determined by $n-1$ points R_i $(i=1, 2, \cdots, n-1)$; as the point P describes a curve C of the hypersurface, each one of the points R_i describes also a curve C_i; if, for any i and j, the tangents at the curves C_i, C_j at R_i, R_j respectively, intersect, the space l is harmonic to the hypersurface. Also the well-known Darboux quadrics are here introduced in the same way as for the surfaces; and other geometrical ideas are presented; we quote a kind of Frenet formulae valid for non-asymptotic curves. The author uses in his paper a covariant derivation, with respect to an affine connection, and thus his computations are very much simplified. All the results appear to be presented as original; the reviewer wishes to point out, however, that no reference is made to papers by Enea Bortolotti [e.g. Boll. Un. Mat. Ital. 6, 134–137 (1927); 7, 87–94, 178–184 (1928); Rend. Sem. Fac. Sci. Univ. Cagliari 1, 38–44 (1931)], where the problems of hypersurfaces in projective S_{n+1} are considered, and a covariant derivation is employed. *V. Dalla Volta.*

Hesselbach, Benno. Konstruktion eines euklidischen E^3 in einem konformen C^4. Math. Nachr. 8, 171–178 (1952).

I. Let x^ν $(\nu, \lambda, \mu=0', 1', 2', 3')$ be a coördinate system of a V_4 with the metric tensor $g_{\lambda\mu}$. Let x^a $(a, b, c=0, 1, 2, 3)$ be another coördinate system. Then we have

$$(1)\qquad I=g_{\lambda\mu}dx^\lambda dx^\mu=g_{ab}dx^a dx^b.$$

In the coördinate system x^ν the expressions $A_a^{\ \nu}=\partial x^\nu/\partial x^a$ $(A_\lambda^{\ a}=\partial x^a/\partial x^\lambda)$ are components of four linearly independent contravariant (covariant) vectors.

II. Consider a pencil of linear hyperspaces whose two elements are the hyperspace of $A_\lambda^{\ 0}$ and the hyperspace perpendicular to $A_0^{\ \nu}$. The equation of this pencil is obviously

$$(2)\qquad P\equiv dx^0(b+g_{00})+g_{0u}dx^u=0,\quad u, v, w=1, 2, 3,$$

where b is the parameter of the pencil. Substituting for dx^0 from (2) into (1) one obtains in P

$$(3a)\qquad I\equiv G_{uv}dx^u dx^v$$

where

$$G_{uv} \equiv g_{uv} - r g_{u0} g_{v0}, \quad r \equiv \frac{2b+g_{00}}{(b+g_{00})^2}.$$

III. Assume the signature of $g_{\lambda\mu}$ to be $-+++$. If we require

$$\text{(3b)} \qquad G_{uv} = m\delta_{uv}, \quad m>0,$$

then the elimination of r from (3) for $u \neq v$ leads to

$$\text{(4)} \qquad g_{12}g_{30} = g_{23}g_{10} = g_{31}g_{20}$$

while the remaining equations reduce by virtue of (4) to three equations

$$\text{(5)} \qquad m = g_{uu} - \frac{g_{uv}g_{uw}}{g_{vw}} \quad (u, v, w \text{ unequal}) \text{ (no sum)}.$$

The equations (4), (5) constitute a system of 5 differential equations for five unknown functions x^a and m which satisfy (3). If we require $F \equiv \sum_{u=1}^{3} g_{uu} - 3m > 0$, the equations (4) and (5) show that there are four numbers such that

$$\text{(6)} \qquad g_{uv} - m\delta_{uv} = q_u q_v, \quad \sum q_v^2 > 0.$$

Putting $g_{00} - q_0^2 = -p^2$ we obtain from (1) and (6)

$$\text{(7)} \qquad I \equiv -p^2(dx^0)^2 + m\delta_{uv}dx^u dx^v + (q_a dx^a)^2.$$

An easy computation shows $b = p(p+\epsilon q_0)$, $\epsilon = \pm 1$, so that (7) reduces for (2) to (3ab). Finally the equations (4), (5), and their solutions are discussed. (The author uses a symbolism of his own writing; for instance, $\mathfrak{s}$, $\bar{x}$, $\bar{y}$, $\bar{z}$ for A_λ^a and so on. This review is transcribed in the usual symbolism.)

V. Hlavatý (Bloomington, Ind.).

Kanitani, Jôyô. Sur les surfaces osculatrices à un espace à connexion projective majorante. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 26, 189–198 (1951).

Suppose that R be a space of n dimensions generated by a point x^i ($i=1, \cdots, n$), and that any curve in R be developed into a curve in a projective space S_N of N dimensions ($N>n$) by means of a connection. Necessary and sufficient conditions are obtained for the existence in S_N of an n-dimensional surface V_n having a contact of a general order $\nu(\geqq 3)$ at the image of the point x^i with the development of any curve in R issuing from the point x^i. The special cases where $\nu=3$, 4, 5 have been discussed in a previous paper of the author [Jap. J. Math. 19, 343–361 (1947); these Rev. 11, 54].

C. C. Hsiung (Bethlehem, Pa.).

Takano, Kazuo. On the two-dimensional surfaces representing the paths in generalized projective space. Rep. Univ. Electro-Commun. 1950, no. 1, 71–81 (1950).

Let an n-dimensional generalized projective space P_n of Veblen be represented by an $(n+1)$-dimensional generalized affine space A_{n+1} in such a way that a point in P_n is represented by a ray of A_{n+1}, and a path of P_n by a two-dimensional surface A_2 generated by the rays which intersect a subpath of A_{n+1}. It is proved that A_2 is totally geodesic in A_{n+1}, and that the induced affine connection Λ for A_2 is symmetric and has no curvature. There are also obtained parameters of A_2 for which all the components of Λ vanish.

C. C. Hsiung (Bethlehem, Pa.).

Egorov, I. P. Collineations of projectively connected spaces. Doklady Akad. Nauk SSSR (N.S.) 80, 709–712 (1951). (Russian)

Let $\Gamma^\alpha_{\beta\gamma}$ be the object of an affine connection in the space X_n. Let $\Pi^\alpha_{\beta\gamma}$ ($x^1, x^2, \cdots, x^n$) be the object of Thomas determining a projective connection in X_n. The components V^α of the local collineation of the projective space X_n are given by the equations

$$\text{(1)} \qquad V^\alpha{}_{,\beta} = U_\beta{}^\alpha,$$

$$\text{(2)} \qquad U^\alpha{}_{\beta,\gamma} = R^\alpha{}_{\beta\gamma\sigma}V^\sigma + \delta_\beta{}^\alpha W_\gamma + \delta_\gamma{}^\alpha W_\beta,$$

where $R^\alpha{}_{\beta\gamma\delta}$ is the tensor of curvature with respect to the connection $\Gamma^\alpha_{\beta\gamma}$ and $W_\alpha(n+1) = V^\sigma{}_{,\sigma\alpha} - R^\sigma{}_{\sigma\alpha\tau}V^\tau$. The conditions of integrability of the system (1) and (2) with respect to the unknowns V^α, W_α, $U_\beta{}^\alpha$ are the following

$$\text{(3)} \qquad D_L W^\alpha{}_{\beta\gamma\delta} = 0,$$

where $W^\alpha{}_{\beta\gamma\delta}$ is the tensor of Weyl and D_L the symbol of Lie derivation and

$$\text{(4)} \qquad (D_L W^\alpha{}_{\beta\gamma\delta})_{,\epsilon} = 0.$$

Further, using equations (3) and (4), the author deduces for $n \geqq 4$ the following relation

$$\text{(5)} \qquad W^{\alpha_1}_{\alpha_2\alpha_3\alpha_4} = 0.$$

A space with projective connection, where $\Pi^1{}_{23} = x^2$ and $\Pi^\alpha{}_{\beta\gamma} = 0$ for the rest of indices, admits the following group of collineations

$$\begin{gathered} p_1, p_3, p_4, \cdots, p_n \\ x^2p_1, x^3p_1, p_2 - x^2x^3p_1, x^2p_2 + 2x^1p_1 \\ x^2p_3 - \tfrac{1}{3}(x^2)^3p_1, \quad x^3p_3 + x^1p_1 \\ x^ip_1, x^2p_i, x^3p_i, x^ip_j \quad (i, j = 4, 5, \cdots, n) \end{gathered}$$

of order n^2-2n+5.

The author proves the following theorems: 1) If X_n is a space with projective connection whose tensor of Weyl is different from zero, then all groups of collineation of maximum order are transitive and of order n^2-2n+5. 2) The maximum order of a group of motions in a space with an affine connection, different from a projectively euclidean space, is n^2-2n+5. All these groups are transitive. 3) Let S_1 be the space with the affine connection $\Lambda^\alpha{}_{(\beta\gamma)}$ corresponding to the space S with the affine connection $\Lambda^\alpha{}_{\beta\gamma}$ which is in general non-symmetric or half-symmetric. If S has the maximal group of motions, then S_1 is a projectively-euclidean space.

F. Vyčichlo (Prague).

Storchi, Edoardo. Integrazione delle equazioni di Codazzi in forma geodetica. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 15(84), 181–184 (1951).

The author considers the equations which express the vanishing of the divergence of a second order tensor in two dimensions. The equilibrium relations satisfied by the stress tensor in the two-dimensional elasticity and plasticity problems are of this type; in differential geometry, the Codazzi relations are of this same type. To integrate the Codazzi relations, the author introduces geodesic coordinates and determines the components of the second fundamental tensor in terms of the metric coefficients.

N. Coburn (Ann Arbor, Mich.).

✱Segre, Beniamino. Forme differenziali e loro integrali. Vol. I. Calcolo algebrico esterno e proprietà differenziali locali. Edizioni Universitarie Docet, Roma, 1951. 520 pp. 3900 Lire.

The applications of the theory of differential forms and their integrals to various branches of mathematics are now so numerous that there is a clear need for a unified account of all aspects of this theory. This volume forms the first part of a work devoted to this end. Its subject is the local theory of forms, with applications.

The first chapter gives an account of Grassmann algebras over a commutative field. After an account of the basic

operations of the algebra, the notion of the adjoint of a form (homogeneous element of the algebra) is introduced, analogous to the dual (*) of a differential form as defined in harmonic integral theory, and by means of it the "regressive product" $*[*\omega_r \cap *\omega_s]$ of two forms ω_r, ω_s of degrees r and s is defined when $r+s \geqq n$. This is used extensively in the classification of r-forms and in establishing inequalities for the rank of a form. The geometrical interpretation of forms plays a large part in this chapter, and many of the proofs are geometrical.

The second chapter introduces differential forms, and develops their calculus and the first properties of their integrals, such as Stokes' theorem. The notion of a harmonic form (for a Euclidean metric) is also introduced. The third chapter deals with certain applications of differential forms, such as the theory of Pfaffian systems, and of invariant integrals under continuous groups of transformations.

The fourth chapter collects a number of important and useful properties of functions of several complex variables, such as properties of analytic subspaces and subvarieties of complex Euclidean space, the notion of residues of multiple integrals. It also deals with some functions of hypercomplex variables, and gives an account of Volterra's theory of conjugate functionals and harmonic integrals.

Each chapter concludes with an extensive bibliography. A proper assessment of the importance and usefulness of the work must await its completion. *W. V. D. Hodge.*

Segre, Beniamino. Alcune applicazioni del calcolo esterno. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 234–239 (1952).

This note applies the methods of Grassmann algebra and the theory of exterior forms to obtain a number of miscellaneous results concerning skew-symmetric matrices, canonical forms for exterior 3-forms in six variables, invariant integrals in Hamiltonian dynamics, and Maxwell's equations in electromagnetism. *W. V. D. Hodge.*

NUMERICAL AND GRAPHICAL METHODS

Rutishauser, Heinz. Automatische Rechenplanfertigung bei programmgesteuerten Rechenmaschinen. Z. Angew. Math. Physik 3, 312–313 (1952).

Burke, Paul J. IBM computation of sums of products for positive and negative numbers. Psychometrika 17, 231–233 (1952).

Teichroew, D. Use of continued fractions in high speed computing. Math. Tables and Other Aids to Computation 6, 127–133 (1952).

Smolyakov, P. T., and Hovanskiĭ, A. N. On the solution of algebraic equations of the 3rd degree. Izvestiya Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk 1, 85–92 (1948). (Russian)

The general cubic is reduced to the form

$$y^3+Py+1=0,$$

and small table to two decimals gives one real root y as a function of P with $|P| \leqq 10$. Besides the table one may use truncated developments of y as a function of P. These are illustrated by examples. *D. H. Lehmer.*

Olver, F. W. J. The evaluation of zeros of high-degree polynomials. Philos. Trans. Roy. Soc. London. Ser. A. 244, 385–415 (1952).

This useful paper forms an excellent summary and critical discussion of numerical methods available for computing zeros of polynomials of degree greater than six, and examines carefully the special problems raised because the degree is large. It is based on considerable practical experience for equations of degrees up to twenty-four. The paper is divided into three parts: Part A deals with direct methods, all basically equivalent to one of the following: (i) Newton's rule, (ii) the root-squaring process, or (iii) the Aitken-Bernoulli process; Part B is concerned with indirect, iterative methods, and Part C with miscellaneous topics, e.g., rounding-off errors and checking. Numerical examples are supplied.

The author's view-point is that of a user of desk machines; automatic machine users should also consult R. A. Brooker [Proc. Cambridge Philos. Soc. 48, 255–270 (1952); these Rev. 13, 691] which gives a very similar and complementary survey from the view-point of an electronic machine user. *J. C. P. Miller* (Cambridge, England).

Aitken, A. C. Studies in practical mathematics. VII. On the theory of methods of factorizing polynomials by iterated division. Proc. Roy. Soc. Edinburgh. Sect. A. 63, 326–335 (1952).

Aitkin [same Proc. 63, 174–191 (1951); these Rev. 12, 860] examined the theory of a method of S. N. Lin [J. Math. Physics 20, 231–242 (1941); these Rev. 3, 153] for the approximation by repeated division to an exact factor of a given polynomial. It is here shown that there are many such methods, and a comprehensive theory for these methods is given. *E. Frank* (Chicago, Ill.).

Duncan, W. J. Note on a generalization of Rayleigh's principle. Quart. J. Mech. Appl. Math. 5, 93–96 (1952).

Let $A(\lambda)x \equiv (A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_0)x = 0$, where the A_p are symmetric square matrices of order n, be the characteristic equation of a set of n linear, homogeneous ordinary differential equations with constant coefficients and of order m. Let λ_s be any one of the mn zeros of $\det A(\lambda)$. Let the corresponding modal column be x_s; thus $A(\lambda_s)x_s = 0$. Then λ_s is one of the m zeros of

$$(*) \qquad \lambda^m x_s' A_m x_s + \lambda^{m-1} x_s' A_{m-1} x_s + \cdots + x_s' A_0 x_s;$$

the other $m-1$ zeros of (*) are irrelevant. It is shown that, if λ_s is a root of (*) of multiplicity $p=1$, first-order errors in x_s lead to second-order errors in the root λ_s of (*). The latter property fails for the irrelevant roots of (*) and also for roots of multiplicity $p>1$. *G. E. Forsythe.*

Łopuszanski, Jan. Solution of Thomas-Fermi equation for molecules with axial symmetry. Acta Phys. Polonica 10, 213–222 (1951).

The method is illustrated by giving the solutions both near to the nuclei and far from the nuclei in the case of a diatomic molecule with two identical atoms. The solutions are the usual spherical ones plus corrections expanded in Legendre polynomials. It is explained how the gap between these solutions can be bridged, and the constants in the series evaluated, by means of interpolation polynomials. *T. E. Hull* (Vancouver, B. C.).

Bechert, Karl. Über ein Verfahren zur näherungsweisen Integration beliebiger partieller Differentialgleichungen. Math. Nachr. 8, 75–78 (1952).

Given an initial value problem for a function $u(x, t)$ satisfying a parabolic or a hyperbolic partial differential equation, the author shows how to obtain a formula for the slope of the curve $u(x, t)=\text{constant}$. Since the initial values are known, the family of curves $u(x, t)=\text{constant}$ can be determined by a stepwise procedure and thus the solution of the partial differential equation is obtained. The reviewer cannot understand the author's remarks about the initial value problem for elliptic type partial differential equation and about the application of the method to the case where grad $u=0$ initially. *B. Friedman* (New York, N. Y.).

Panov, D. Yu. On approximate numerical solution of quasilinear partial differential equations of hyperbolic type. Doklady Akad. Nauk SSSR (N.S.) 83, 793–795 (1952). (Russian)

Panov, D. Yu. On making more precise the values of the unknowns in approximate numerical solution of quasilinear partial differential equations of hyperbolic type. Doklady Akad. Nauk SSSR (N.S.) 84, 17–19 (1952). (Russian)

In the first of these two papers the author proposes a method for solving certain partial differential equations of hyperbolic type by calculating the coordinates of the nodes of a net formed by selected curves from the two families of characteristics. Thus instead of the usual practice of taking a rectangular net of lines parallel to the x- and y-axes and finding the values of the dependent variables at the nodes, the new idea is to construct a curvilinear net made up of characteristic curves. Assuming certain points already known, the author shows how to calculate approximately the coordinates of new nodes. The second paper provides special formulas of mechanical quadrature adapted to the problem of making more precise the results obtained in the first approximation. *W. E. Milne* (Corvallis, Ore.).

Mitrović, D., and Tomović, R. Solution of the partial differential equation of the heat-flow on the a. c. network analyser. Srpska Akad. Nauka. Zbornik Radova, Knj. 18, Matematički Inst., Knj. 2, 181–186 (1952). (Serbo-Croatian. English summary)

Holms, Arthur G. A biharmonic relaxation method for calculating thermal stress in cooled irregular cylinders. NACA Rep. no. 1059, 19 pp. (1952).

✱Bückner, H. Die praktische Behandlung von Integral-Gleichungen. Ergebnisse der angewandten Mathematik. Bd. 1. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1952. vi+127 pp. DM 18.60.

This text provides a concise résumé of most of the known methods for the approximate treatment of one-dimensional non-singular Fredholm integral equations of the second kind, and includes much material which has not previously appeared in integrated form. Derivations and proofs are included in most cases; otherwise, specific references are made to the sources. Of the five chapters included, the first is introductory, the second deals with variational principles, the third with iterative techniques, the fourth with perturbation methods, and the fifth with techniques relevant to kernels of special type. Twenty numerical examples of simple nature are included. The usefulness of the text is enhanced by a list of ninety-three references, but is rather seriously impaired by the brevity of the index.
F. B. Hildebrand (Cambridge, Mass.).

Weissinger, J. Über die Einschaltung zusätzlicher Punkte beim Verfahren von Multhopp. Ing.-Arch. 20, 163–165 (1952).

The Trefftz-Glauert procedure for obtaining an approximate solution of the Prandtl lifting-line equation consists in determining the coefficients of an n-term truncated Fourier series by collocation at n points. The Multhopp technique, which is applicable only when these points are equally spaced, and advantageous only when direct solution of the resultant set of n linear algebraic equations is inconvenient, amounts to transforming that set into an equivalent set which is generally more amenable to Gauss-Seidel iteration. In the present paper it is shown how the transformed set of equations can be obtained when one additional harmonic is introduced, together with an additional arbitrary point of collocation. [Since the new set of equations may not be suitable for iterative solution, the advantage of the Multhopp transformation in such a case appears to be limited.]
F. B. Hildebrand (Cambridge, Mass.).

Weissinger, J. Die Auftriebsverteilung von Tragflügeln mit Tiefensprung. Ing.-Arch. 20, 166–169 (1952).

The modified Multhopp transformation described in the preceding review is applied to a set of equations previously derived by the reviewer [NACA Tech. Note no. 925 (1944); these Rev. 7, 220] in connection with the approximate solution of the Prandtl lifting-line equation relevant to an airfoil with discontinuous chord variation. [The author's remark to the effect that the labor involved in the transformation is less than that involved in a certain least-squares technique is misleading, since the approximate lift distribution corresponding to his transformed set of equations is identical with that corresponding to the set obtained directly by collocation, whereas the least-squares technique would be used only if a more accurate approximation were required.]
F. B. Hildebrand (Cambridge, Mass.).

Griffith, B. A., and Smillie, K. W. On a punched-card method of solving certain integral equations. Math. Tables and Other Aids to Computation 6, 133–138 (1952).

Tasny-Tschiassny, L., and Doe, A. G. The solution of polynomial equations with the aid of the electrolytic tank. Australian J. Sci. Research. Ser. A. 4, 231–257 (1951).

Any physical device which approximately locates the roots of a polynomial with complex coefficients can be used to obtain the successive approximations needed to apply Horner's method for calculation of complex roots. This paper is not concerned with giving efficient numerical procedures for effecting the successive changes of origin required by Horner's method but rather with the technical details for the practical realization of A. Bloch's suggestion [Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 4, p. 324; these Rev. 11, 403] that sinusoidal current sources can be used to extend the method of Lucas [C. R. Acad. Sci. Paris 106, 645–648 (1888)] to complex polynomials. The transformation $w=-d/(z+i)$ is used to map the upper half-plane into a circular tank of diameter d. *R. Church.*

Hersom, S. E., and Selig, K. L. A general purpose differential analyser. II. Application of machine. Elliott J. 1, 76–80 (1952).

For part I see Ashdown and Selig, same vol., 44–48 (1948); these Rev. 13, 592.

Tomovich, Rajko. A universal unit for the electrical differential analyzer. J. Franklin Inst. 254, 143–151 (1952).

Abbott, Wilton R. Computing logical truth with the California Digital Computer. Math. Tables and Other Aids to Computation 5, 120–128 (1951).

Hargest, T. J. An electric tank for the determination of theoretical velocity distributions. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2699 (12,448), 9 pp. (1952).

Huard de la Marre, Pierre. Étude rhéoélectrique de problèmes d'infiltration. C. R. Acad. Sci. Paris 235, 602–603 (1952).

Benscoter, Stanley U., and MacNeal, Richard H. Introduction to electrical-circuit analogies for beam analysis. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2785, 48 pp. (1952).

Urcelay, Jose Maria. Rhomboidal nomograms. Gaceta Mat. 3, 183–194 (1951). (Spanish)

Three vertices of a parallelogram are taken from curve nets and the fourth is from a family of curves to yield a type of nomogram considered by the author to be superior to the equivalent representation possible by superimposed planes capable of relative translation. *R. Church.*

ASTRONOMY

Grant, Fraser S. Three dimensional interpretation of gravitational anomalies. Geophysics 17, 344–364 (1952).

The possibilities of mathematical methods in the interpretation of gravimetric and magnetic anomalies are now attracting more and more the attention of researchers. The method developed in this article belongs to the class of mathematical methods. In it the determination of the approximate size, shape and depth of a three-dimensional mass-distribution generating a given gravitational field is based on computation from the anomaly map of the first few reduced multipole moments of distribution, provided the density-contrast is known. A new feature of this method consists in the introduction of a classification of three-dimensional mass-distributions into four principal groups by symmetry. In each of these four groups the non-vanishing moments are the same. The four groups are: cubic (cube and sphere), tetragonal (tetragonal block, circular cylinder, and ellipsoid of revolution), orthorhombic (rectangular block, elliptical cylinder, and ellipsoid), and monoclinic (trapezoidal block, rhombic block, and symmetrical wedge). The practical importance of such interpretation methods as the author's method in the hands of a geophysicist possessing the necessary mathematical background is enormous.
E. Kogbetliantz (New York, N. Y.).

Magnaradze, N. G. On the convergence of the expansions of the Newtonian potential of an elliptic orbit in certain boundary points of the region of convergence. Akad. Nauk Gruzin. SSR. Abastuman. Astrofiz. Obs. Byull. 11, 143–153 (1950). (Georgian. Russian summary)

The aim of the present work is to prove the convergence of the infinite expansions of the Newtonian potential of an elliptic orbit in certain boundary points of the region of convergence; the expansions were determined in another paper [Trudy Tbiliss. Gos. Univ. 40, 1–35 (1950); unavailable].
From the author's summary.

Magnaradze, N. G. On an estimate of the remainder terms of the expansions of the Newtonian potential of an elliptic orbit. Akad. Nauk Gruzin. SSR. Abastuman. Astrofiz. Obs. Byull. 11, 155–161 (1950). (Georgian. Russian summary)

In this paper estimates are given of the remainder terms of the infinite expansions of the Newtonian potential of an elliptic orbit established by us in another paper [reference in preceding review]. *From the author's summary.*

Fleckenstein, J. O. Les théorèmes de Laplace sur les perturbations séculaires dans les éléments vectoriels des orbites planétaires. Experientia 8, 136–137 (1952).

This investigation deals with a proof of Laplace's theorem of the invariability of the major axes of perturbed planetary orbits. Equations of motion in vectorial elements, introduced by Milankovitch, are used. There is a misprint in the first two of the equations (5), in which the first terms are interchanged. The last of the equations (5),

$$\frac{d}{dt}\left(\frac{\mu^2-D^2}{C^2}\right)=2\frac{\partial R}{\partial t},$$

is identical with the well-known equation

$$\frac{da}{dt}=\frac{2}{au}\frac{\partial R}{\partial t}.$$

Now the author states that, because $-\frac{1}{2}((\mu^2-D^2)/C^2)$ is equal to the total energy of the perturbed planet, this quantity is free of secular changes. From this follows his Lemma I that R is secular independent of the time of perihelion t. Then the relation between $(\mu^2-D^2)/C^2$ and the major axes gives Laplace's theorem. It is obvious, however, that the author's first statement is identical with Laplace's theorem, so that no actual proof is given.
A. J. J. van Woerkom (New Haven, Conn.).

Bucerius, H. Bahnbestimmung als Randwertproblem. III. Astr. Nachr. 280, 73–82 (1951).

Continuing a program described in two previous papers [Astr. Nachr. 278, 193–203, 204–216 (1950); these Rev. 12, 753], the author studies the question of uniqueness of the solution of the equations of the Gauss method for orbit determination. *W. Kaplan* (Ann Arbor, Mich.).

Andersson, Sven. On the change with time in the disturbed motion of two bodies. Ark. Astronomi 1, 207–214 (1952).

In the two-body problem the motion of one body relative to the other is in a conic section for any initial conditions. In the disturbed motion of the two bodies it is convenient to suppose that the one moves relative to the other in a conic section whose position and form are constantly changing. The author gives a direct proof of the formulas expressing the time rate of change of the elements in the instantaneous conic section. *E. Leimanis* (Vancouver, B. C.).

Schmeidler, F. Intermediäre Bahnen zur Annäherung an das Dreikörperproblem. Astr. Nachr. **280**, 245–253 (1952).

The elliptic orbit as an intermediate orbit for the determination of planetary motions originated with Newton and his successors. The author proposes to start, not with an ellipse, but with an intermediate orbit which gives a better approximation to the problem than the undisturbed elliptic motion. In particular, an intermediate orbit may include terms of the perturbation function chosen in such a way as to make the perturbation problem obtained in this way integrable. Using the theorem that a system of differential equations of order $2n$ is integrable if n integrals whose Poisson brackets vanish are known, the author derives two particular intermediate orbits which take into account some kinds of perturbation terms. One of these includes the secular terms, the other includes all terms of the perturbation function which are independent of eccentricities and inclinations. In the latter case the solution in powers of the disturbing mass leads to elliptic integrals. *E. Leimanis.*

Sorokin, V. S. Investigation of equilibrium and stability of isothermic gaseous spheres. Akad. Nauk SSSR. Astr. Žurnal **29**, 25–36 (1952). (Russian)

The object of the author's investigation is to analyze the structure of self-gravitating isothermal spheres consisting of partially degenerate gas. The degeneracy considered is non-relativistic; and the second-order differential equation whose solution expresses the distribution of internal density and pressure is set up in terms of the Fermi-Dirac function, converted to a nondimensional form [for its derivation cf. Chandrasekhar, An introduction to the study of stellar structure, Univ. of Chicago Press, 1939, pp. 447–448], and solved numerically for seven different initial values of $\eta=\chi/kT$ ranging from -2 to 20. These solutions are partially tabulated, and the variation of η with increasing distance from the center illustrated by a diagram.

The most interesting result of this investigation is the fact that, with increasing degree of degeneracy, the mass of an isothermal gas sphere tends to a finite limit, in contrast with ideal-gas isothermal spheres which are known to be of infinite masses and dimensions. Now degenerate configurations are perhaps the only types of objects in nature which are likely to approach isothermal state (since the thermal conductivity of a degenerate gas is so high as to eliminate effectively any appreciable internal temperature gradient); but for temperatures of the order of 10^7 deg (encountered in stellar interiors), the masses of Sorokin's configurations turn out to be so high (comparable, in fact, with the mass of the Sun), that relativistic effects are no longer negligible and must be considered before the results may serve for astrophysical applications. *Z. Kopal* (Manchester).

Feigel'son, E. M. Radiative properties of clouds St. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. **1951**, no. 4, 92–117 (1951). (Russian)

The author considers the problem of scattering of sunlight in atmospheric clouds, and embarks for this purpose on the construction of a solution of the well-known equation of transfer (in a plane-parallel approximation) of incident radiation scattered in accordance with an indicatrix expansible in terms of the Legendre polynomials $P_n(\theta)$ of the angle θ between the incident and scattered beams. He terminates this expansion with $n=2$, and attempts a solution of so simplified an equation of transfer by the method of Kuznetsov [same no., 71–91 (1951); these Rev. **13**, 248], assuming the solution to terminate with the second harmonic of the azimuth angle. The solution has been worked out numerically, and used for an evaluation of the center-to-limb variation of brightness of the cloud, of its coefficient of transmission as well as of the albedo in the light of long wave lengths. *Z. Kopal* (Manchester).

Ževakin, S. A. Discrete model of a star (oscillation and explosion). I. Akad. Nauk SSSR. Astr. Žurnal **29**, 37–48 (1952). (Russian)

The author sets out to investigate the way in which stellar models, in radiative equilibrium, in the interiors of which the absorption coefficient varies as a certain (constant) power of the temperature and density can perform free radial oscillations; and concludes that the temperature-dependence of thermonuclear reactions cannot, in general, be the cause of their pulsation. *Z. Kopal* (Manchester).

Chatterji, L. D. Anharmonic pulsations of a polytropic model of index unity. Proc. Nat. Inst. Sci. India **18**, 187–191 (1952).

It is found for this model that the first overtone contributes a skewness to the velocity-time curve in the right direction for the Cepheids but of an amount smaller than that observed. *R. G. Langebartel* (Urbana, Ill.).

Bondi, H. On spherically symmetrical accretion. Monthly Not. Roy. Astr. Soc. **112**, 195–204 (1952).

A star of mass M is at rest in an infinite cloud of gas, which at infinity is also at rest and of uniform density ρ_∞ and pressure p_∞. The motion of the gas is spherically symmetrical and steady, and in the gas $p/p_\infty=(\rho/\rho_\infty)^\gamma$, $1\leqq\gamma\leqq 5/3$. The accretion rate is then proportional to $M^2c^{-3}\rho_\infty$, where c is the velocity of sound in the gas at infinity. *R. G. Langebartel* (Urbana, Ill.).

RELATIVITY

***Møller, C. The theory of relativity.** Oxford, at the Clarendon Press, 1952. xii+386 pp. $7.00.

This monograph in the International Series is intended as a textbook for students in physics whose mathematical and physical training does not go beyond the methods of non-relativistic mechanics and electrodynamics. As such, it is devoted entirely to an account of the role of the special theory of relativity in electrodynamics and mechanics, including elasticity and thermodynamics, and to the general relativistic theory of gravitation. No attempt is made to treat quantum phenomena or cosmology, except for a brief account of the inadequate Einstein and de Sitter cosmological models. Aspects of differential geometry and tensor calculus are developed in the text, and a number of auxiliary topics, such as Gauss' theorem and action principles, are relegated to appendices.

The book opens with an excellent historical account of the optical effects which led to the development of the special theory of relativity. It then goes on to develop relativistic kinematics and mechanics in terms of three-dimensional vector calculus, without reference to the four-dimensional picture. This latter is then presented in Chap.

IV, but in terms of imaginary time *ict* and without employing the useful and illuminating Minkowski diagram. These chapters contain an extended treatment of the most general Lorentz transformations and their composition. Electrodynamics in vacuo and in ponderable media are quite thoroughly discussed in Chaps. V and VII, between which is a chapter on closed mechanical systems. All in all, the treatment of special relativity is fuller and more satisfactory than that found in most modern textbooks on relativity—although the reviewer must admit to a prejudice in favor of the real four-dimensional representation, in which the mass of an elementary particle is an inherent and invariant measure of its inertia, independent of the accidental motion of an observer relative to it!

The general theory of relativity does not fare so well in Møller's treatment. Much stress is laid on the so-called principle of equivalence, and on an artificial and cumbersome breakdown of space-time into space and time. This part of the text opens with a chapter on the equivalence principle and the spurious kinematic forces resulting from time-dependent transformations of the special relativity line element. This is followed by two chapters on permanent (as opposed to kinematic) gravitational fields, and their influence on physical phenomena. Chap. XI sets up the field equations connecting the metric of space-time with its physical content, and solves them for the case of weak fields and for the general spherically symmetric field (Schwarzschild). The book closes with a short chapter on the three observational tests of the theory in the field of the sun, and a brief account of the Einstein and de Sitter cosmological models. *H. P. Robertson* (Pasadena, Calif.).

***Ludwig, G. Fortschritte der projektiven Relativitätstheorie.** Friedr. Vieweg & Sohn, Braunschweig, 1951. 96 pp.

The first part of this book contains a mathematical exposition of Jordan's generalization of Veblen's projective relativity. This generalization consists of considering a certain scalar J as a function of coordinates instead of a constant. Later this scalar is interpreted as a generalization of Einstein's gravitational constant. The author gives a clear exposition of the principles of projective relativity and describes a formalism for handling this theory expeditiously. The connections between five-dimensional projective tensors and the usual four-dimensional tensors are given in detail. This part of the book ends with a discussion of properties of field equations derived from variational principles.

The second half of the book is devoted to a discussion of the physical interpretation of the theory when the matter generating the gravitational field is described by: (1) a stress-energy-tensor for a perfect fluid; (2) a Lagrangean function depending on a scalar field; and (3) a Lagrangean function depending on a spinor field.

In the discussion labeled (1) the author describes briefly Jordan's cosmological model in which J varies as t^{-1} where t is cosmic time and Jordan's theory of the origin of stars in which J varies as t^{-2}. These two theories result from two solutions of the field equations derived from action principles in which certain arbitrary functions of J are first introduced and then given simple forms, that is, the field equations involved are in a certain sense arbitrary. This arbitrariness is still in the theory when it is formulated to be invariant under the projective relativity group. One would have hoped that the five-dimensional projective relativity formulation of the theory would have led to the field equations considered in a more natural manner.

A. H. Taub (Urbana, Ill.).

Hlavatý, Václav. The elementary basic principles of the unified theory of relativity. Proc. Nat. Acad. Sci. U. S. A. 38, 243–247 (1952).

The unified theory of Einstein is based on (A) the introduction of a non-symmetric tensor $g_{\lambda\mu}$; (B) the introduction of a non-symmetric connexion by means of

$$\nabla_\nu g_{\underset{+}{\lambda}\underset{-}{\mu}}=0$$

and (C) the introduction of a system of conditions for the $\Gamma^\nu_{\mu\lambda}$. In this note some results are given with respect to each of these points. As to (A) it is pointed out that the non-symmetric fundamental tensor can be connected easily with spin theory. (B) leads to sets of equations that simplify in the case of an almost Riemannian space. The conditions (C) are compared with the first and second Einstein conditions. Proofs are to be published later. *J. A. Schouten* (Epe).

Tonnelat, Marie-Antoinette. Compléments à la théorie unitaire des champs. J. Phys. Radium (8) 13, 177–185 (1952).

In this paper the author collects results on and compares different unified field theories each of which is based on a non-symmetric affine connection and involves a non-symmetric tensor $g_{\mu\nu}$ whose symmetric part represents gravitation. The field equations are assumed to be derivable from a variational principle. The author proposes as a Lagrangean a quantity proportional to the square root of the determinant of the contracted curvature tensor. The field equations obtained are discussed in terms of an approximation in which the size of the symmetric part of $g_{\mu\nu}$ is unrestricted but the anti-symmetric part is assumed to be of order ϵ and terms in ϵ^2 are neglected. The resulting equations for the electromagnetic field are shown to differ from the Maxwell equations. *A. H. Taub* (Urbana, Ill.).

Weyssenhoff, Jan, and Raabe, A. Relativistic dynamics of spin-fluids and spin-particles. Acta Phys. Polonica 9, 7–18 (1947).

Each element of a spin-fluid possesses, in addition to energy and momentum, an amount of angular momentum proportional to the volume of the element. This paper is an introduction to the classical special relativistic theory of such fluids, giving a simpler method of deriving the equations of motion in the presence of an electromagnetic field. The connection is established between the asymmetry of the energy-momentum tensor and the existence of intrinsic spin, and the integral over a small volume of the scalar product of the velocity and momentum density is shown to be a constant of the motion and is interpreted as the rest-mass. The free motion is a superposition of a uniform velocity and a circular motion, but the radius of the circle can acquire very large values. The equations of motion are of third order. *H. C. Corben* (Genoa).

Weyssenhoff, Jan, and Raabe, A. Relativistic dynamics of spin-particles moving with the velocity of light. Acta Phys. Polonica 9, 19–25 (1947).

The theory of the paper reviewed above is applied to a particle moving with the velocity of light, the direction of motion changing with time. The mass of the particle must here be defined in a different manner. *H. C. Corben*.

Weyssenhoff, Jan. Further contributions to the dynamics of spin-particles moving with a velocity smaller than that of light. Acta Phys. Polonica 9, 26–33 (1947).

The theory of spin-particles of velocity less than that of light is restated by putting the non-relativistic equations in tensor form. *H. C. Corben* (Genoa).

Weyssenhoff, Jan. Further contributions to the dynamics of spin-particles moving with the velocity of light. Acta Phys. Polonica 9, 34–45 (1947).

The equations of motion of spin-particles moving with the velocity of light are written in vector form and the spin-vector is shown to be perpendicular to the velocity and to the acceleration. The constants of the motion when the particle moves in an electromagnetic field are investigated. *H. C. Corben* (Genoa).

Weyssenhoff, Jan. On two relativistic models of Dirac's electron. Acta Phys. Polonica 9, 46–53 (1947).

The analogy between the motion of a spin-particle and a Dirac electron is studied, and the analogy is shown to be more striking when the spin-particle moves with the velocity of light. In particular the mass of such a spin-particle is approximately constant under conditions which for the Dirac theory correspond to the absence of real pair-production. *H. C. Corben* (Genoa).

Weyssenhoff, Jan. Relativistically invariant homogeneous canonical formalism with higher derivatives. Acta Phys. Polonica 11, 49–70 (1951).

The canonical formalism of classical relativity theory is summarized for the case in which higher derivatives appear in the Lagrangian and an arbitrary parameter is introduced as independent variable. Possible choices of the Lagrangian are discussed and compared with the work of other authors and it is shown that the theory of spin-fluids results as a special case. *H. C. Corben* (Genoa).

Średniawa, Bronisław. Relativistic equations of motion of free dipole and quadrupole particles. Acta Phys. Polonica 9, 99–108 (1948).

Mathisson's variational principle [Acta Phys. Polonica 6, 163–200 (1937)] for describing the motion of multipole particles is developed within the framework of classical special relativistic mechanics, and the dipole case is shown to correspond to the motion of spin-particles. The simplest of the other possibilities (dipole-quadrupole particle, quadrupole particle) are investigated. *H. C. Corben* (Genoa).

Majorana, Quirino. Spazio e tempo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12, 481–486 (1952).

Hély, J. La notion de masse et la dynamique du point. Mémorial de l'Artillerie Française 24, 899–941 (1950).

The primary aim of this work is the synthesis of the gravitational and electromagnetic fields in one comprehensive theory, rejecting the general theory of relativity but accounting for the three well-known Einstein effects in planetary and optical motion. In his first attempt at such a synthesis, the author adds to the action of the mechanical and electromagnetic fields a gravitational term suggested by M. Abraham's 1912 special relativistic theory of gravitation. This rather unforced theory is rejected because of its inability to account for the Einstein effects, as well as because of some abstruse criticism levelled at the photon concept implied thereby. The new and final synthesis is obtained by replacing the gravitational term in the action by a curious two-valued one, with the aid of which the author deduces the accepted (general relativity) values for the Einstein effects. The two-valuedness of the gravitational action is rationalized by an appeal to the difference in spin between an elementary particle and a photon. *H. P. Robertson* (Pasadena, Calif.).

Synge, John L. Vitesse de phase et vitesse de groupe en optique relativiste. Rev. Optique 31, 121–122; réponse de Mario Galli, 122–123 (1952).

J. L. Synge gives a short proof of the relation $c_\varphi c_g = c^2$ between phase and group velocities, assuming a Lorentz invariant dispersive law. This seems to contradict an earlier statement of M. Galli [Rev. Optique 30, 174–184 (1951); these Rev. 13, 290]. In his reply, Galli points out that the two authors' interpretations of the term "relativity principle" differs. His own definition includes the equivalence of inertial frames but excludes the constancy of the velocity of light. *A. Schild* (Pittsburgh, Pa.).

Galli, M. Il ruolo dell'etere nell'ottica relativistica generalizzata. Ottica (N.S.) 5, 112–114 (1951).

The author discusses the relationship of the concept of the ether to the optical theory developed in a previous paper [Ottica (N.S.) 5, 49–62 (1951); these Rev. 13, 695], and reaffirms his opinion that the relationship $c_\varphi c_g = c^2$ between phase- and group-velocities, although Lorentz-invariant, is not a consequence of the theory of relativity [cf. M. Galli, Revue d'Optique 30, 174–184 (1951); these Rev. 13, 290; and the paper reviewed above]. *J. L. Synge* (Dublin).

MECHANICS

Hain, K. Zur Synthese der Schiebepaar-Getriebe. Ing.-Arch. 20, 184–188 (1952).

The design of a crosshead mechanism is sought which will produce three given angular rotations of a crank to correspond to three given linear motions of a slide. By graphical constructions, it is possible to determine the length of the crank, its center, the length of the connecting-rod and the location of the slide guide. When the length of the connecting-rod is infinite, it may be replaced by an additional sliding connection. A point carried by a connecting-rod may be made to pass through five selected points. Again, a graphical construction produces the design of the crosshead mechanism. Special cases are considered. For example, the curve may be passed through six arbitrary points which lie, in pairs, on three parallel straight lines. *M. Goldberg.*

Yudin, V. A. Certain questions in the dynamics of mechanisms with higher pairs, allowing for friction. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 10, no. 38, 5–26 (1950). (Russian)

The author is interested in the equation of motion of the (rotary or translatory) driving member of a plane cam mechanism. It will be sufficient here to consider the rotary case. If the equation is written in the usual form, $\frac{1}{2}d(I\dot{\varphi}^2)/dt = M$, where I and M are the "reduced moment of inertia" (variable) and the "reduced torque load". The paper claims, in essence, that the effect of friction can be

accounted for by putting, in the general case of k cams and negligible inertia forces (but not moments),

$$I = I_1 + \sum_{j=2}^{k} I_j (i_{21} i_{32} \cdots i_{j,j-1})^2 / (\eta_1 \eta_2 \cdots \eta_k),$$

and doing the same for the M's (individual applied torques) with the exponent 2 replaced by 1. The i's are the ratios of the instantaneous angular velocities, and the η's "corrective coefficients" equal to unity in absence of friction. This is hard to believe for $k > 2$ because any intermediate cam will be in a condition of indeterminacy, making two point contacts with its neighbors, and one surface contact with its pivot. In fact, the only (approximate) values of η given are for cams with one point contact only (they depend on the friction coefficients and angles of pressure). The proof offered in the paper involves three cams only. On closer inspection of it, one is puzzled by mysterious disappearances of certain reaction forces and moments from Eqs. (13) and (16). This is done by setting a sum of n equal forces equal to zero, etc. The reviewer believes that the formulas given, however attractive, must be regarded as largely ornamental, although they are true in absence of friction ($\eta = 1$).

A. W. Wundheiler (Chicago, Ill.).

Manolov, Spasse. Sur l'existance des petits mouvements périodiques d'une configuration mécanique. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 377–384 (1950). (Bulgarian. French summary)

A double plane pendulum, consisting of two rods of the same length $2a$ is rotating at a uniform angular speed ω around a fixed vertical axis. If $\omega^2 < 3g(7^{1/2}-2)/2a7^{1/2}$, and if δ is sufficiently small, there exist small oscillations about the equilibrium position, and their period is

$$\delta + 2\pi / \{[3g(7^{1/2} \pm 2)/2a7^{1/2}] - \omega^2\}.$$

A. W. Wundheiler (Chicago, Ill.).

Pöschl, Theodor. Eine Bemerkung zu den Beispielen aus der Mechanik. Math. Nachr. **8**, 155–156 (1952).

Christov, Chr. Sur les notions et les lois de la mécanique classique. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **46**, 211–270 (1950). (Bulgarian. French summary)

Essai d'axiomatique assez compliqué et dont l'intérêt est limité par le fait qu'on ne considère que les corps solides et des distributions de masses à densité continue.

M. Brelot (Grenoble).

Müller, Wilhelm. Zur Theorie des Reibungsstosses einer Kugel gegen eine ebene Wand und gegen eine zweite Kugel. Österreich. Ing.-Arch. **6**, 196–208 (1952).

Plane collision problems (sphere with sphere and sphere with plane wall) are treated by simple methods. The results are less general than known results obtained by the method of the representative point [E. J. Routh, Elementary rigid dynamics, 6th ed., Macmillan, London, 1897, p. 158; see p. 165 for the rebound of a sphere from a wall].

J. L. Synge (Dublin).

de Castro Brzezicki, A. On plane motion of rockets. Revista Mat. Hisp.-Amer. (4) **12**, 102–106 (1952). (Spanish)

Perret, Eduard, Roth, Ernst, Sänger, Raymund, und Voellmy, Hans R. Flugbahnen von Leitstrahlraketen mit Gasstrahlsteuerung. Z. Angew. Math. Physik **3**, 241–258 (1952).

Hydrodynamics, Aerodynamics, Acoustics

★Nielsen, Jakob. Laerebog i rationel mekanik. III. Vektoranalyse, potentialteori, kontinuerlige medier, strømninger, komplekst potential. [Textbook in rational mechanics. III. Vector analysis, potential theory, continuous media, flows, complex potential.] Jul. Gjellerups Forlag, Copenhagen, 1952. viii+197 pp.

The first two volumes of this text were subtitled Statik [1933, 1943] and Dynamik [1934, 1945]. This volume, stemming from lectures given at the Danish Technical University and the University of Copenhagen, is designed to provide the necessary mathematical background for a study of the mechanics of continua as well as an introduction to some of its fundamental concepts and theorems. Thus, there is no discussion of problems associated with special geometrical configurations with the exception of a treatment of the Joukowski airfoil at the end of the last chapter and an occasional example. The standards for both rigor and clarity are high and this text should make a valuable companion to standard treatises on the mechanics of continua. Only linear elasticity and viscosity laws are considered. The subtitle above gives the titles to the five chapters and there is also an appendix on dyads.

J. V. Wehausen.

Roseau, Maurice. Ondes liquides de gravité en profondeur variable. C. R. Acad. Sci. Paris **233**, 844–845 (1951).

The problem is that of time-periodic two-dimensional gravity waves of small amplitude in a channel of infinite length, the depth of which tends to different constant values at the two infinities. The transition curve of the bottom is not arbitrary, but contains a number of arbitrary parameters. The problem of progressing waves in such channels is attacked by representing the velocity potential as a sum of two complex integrals of the Laplace type with unknown "kernels" and paths of integration. The two unknown functions under the integral signs are then shown to satisfy difference equations by virtue of the boundary conditions. [This procedure is similar to that followed by Peters [Comm. Pure Appl. Math. **3**, 319–354 (1950); these Rev. **12**, 869] and the author [same C. R. **232**, 211–213, 303–306 (1951); these Rev. **12**, 869] in studying the problem of waves over uniformly sloping beaches.]

J. J. Stoker.

Roseau, Maurice. Résolution d'équations fonctionnelles qui se présentent dans le problème des ondes liquides de gravité en profondeur variable. C. R. Acad. Sci. Paris **233**, 916–917 (1951).

The author indicates the method used to solve the difference equations referred to in the preceding review.

J. J. Stoker (New York, N. Y.).

Roseau, Maurice. Réflexion des ondes dans un canal de profondeur variable. C. R. Acad. Sci. Paris **234**, 297–299 (1952).

Properties of the solutions of the problem discussed in the preceding reviews that are significant from the physical point of view are derived.

J. J. Stoker.

Gröbner, W. Oberflächenwellen von Flüssigkeiten. Ann. Scuola Norm. Super. Pisa (3) **5**, 175–191 (1951).

The author employs Hamilton's principle to obtain approximate solutions of problems concerning surface gravity waves of small amplitude. In setting up the kinetic energy integral, various different types of assumptions are made concerning the vertical (y-direction) distribution of the horizontal velocity component u: (a) u is independent of y; (b) u is linear in y and vanishes at a rigid bottom; (c) u behaves exponentially in y. In each of these cases the author solves one or more problems by integrating the variational equations arising from Hamilton's principle. In case (a), for example, he obtains among other things the well-known results of the linear shallow water theory. Other problems treated include two-dimensional waves over a uniformly sloping bottom, and waves having cylindrical symmetry.

J. J. Stoker (New York, N. Y.).

Liu, Hsien Chih. Über die Entstehung von Ringwellen an einer Flüssigkeitsoberfläche durch unter dieser gelegene, kugelige periodische Quellensysteme. Z. Angew. Math. Mech. **32**, 211–226 (1952). (German. English, French and Russian summaries)

The author finds the velocity potential for the harmonic motion of a heavy liquid with a free surface and a source of pulsating strength situated beneath the surface. The problem is treated three-dimensionally and with the linearized free surface boundary condition. Expansions are derived which are suitable for finding the form of the free surface both in the region over the source and at great distances. Graphs are given for special values of the parameters. Some related papers seem to have been overlooked [e.g., Havelock, Philos. Mag. (7) **33**, 666–673 (1942); these Rev. **4**, 60; Kočin, Učenye Zapiski Moskov. Gos. Univ. Mehanika **46**, 85–106 (1940); these Rev. **12**, 59; John, Comm. Pure Appl. Math. **3**, 45–101 (1950); these Rev. **12**, 214].

J. V. Wehausen (Providence, R. I.).

Packham, B. A. The theory of symmetrical gravity waves of finite amplitude. II. The solitary wave. Proc. Roy. Soc. London. Ser. A. **213**, 238–249 (1952).

Parts I and III of this paper are by Davies [same Proc. Ser. A. **208**, 475–486 (1951); Quart. Appl. Math. **10**, 57–67 (1952); these Rev. **13**, 396, 698]. In this paper the same method is applied to the case of the solitary wave. Only the first approximation (i.e., replacing the boundary condition $\partial\theta/\partial\psi = -gc^{-3}\epsilon^{-3\tau}\sin\theta$ on $\psi=0$ by $\partial\theta/\partial\psi = -glc^{-3}\epsilon^{-3\tau}\sin 3\theta$) is carried through, but this can be solved exactly to give wave velocity and amplitude.

J. V. Wehausen.

Kierstead, Henry A. Bottom pressure fluctuations due to standing waves in a deep, two-layer ocean. Trans. Amer. Geophys. Union **33**, 390–396 (1952).

The author gives a short heuristic proof of some recent results of M. S. Longuet-Higgins [Philos. Trans. Roy. Soc. London. Ser. A. **243**, 1–35 (1950); these Rev. **12**, 763] on the subject of the title and extends the results to a two-layered ocean.

J. V. Wehausen (Providence, R. I.).

Monaghan, R. J. A theoretical examination of the effect of deadrise on wetted area and associated mass in seaplane-water impacts. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2681** (**12,395**), 16 pp. (1952).

Manwell, A. R. Constant velocity aerofoils with circulation. Proc. London Math. Soc. (2) **54**, 168–183 (1952).

Riabouchinsky [same Proc. **19**, 206–215 (1920)] derived airfoil profiles for constant pressure distribution; these are symmetrical profiles at zero lift. Manwell [Quart. J. Mech. Appl. Math. **1**, 365–375 (1948); these Rev. **10**, 411] showed that these are minimum-velocity arifoils for given thickness ratios. Here the Riabouchinsky series is generalized to provide different constant values of the pressure on upper and lower surfaces and thus circulation and lift. The method makes use of the hodograph plane and there is the usual problem of assuring that the profiles are closed. These airfoils with lift do not have the property of minimum velocity for given thickness ratio. This the author proves by considering the special case of an infinitesimally thin member of the series, with infinitesimal lift. He points out that minimum-velocity profiles with lift need not have constant velocity on their lower surfaces, and conjectures that the maximum velocity may occur somewhere on their lower surfaces as well as on their upper surfaces.

W. R. Sears.

Prosciutto, Aristide. Sulle proprietà caratteristiche di particolari tipi di schiere di pale, generate mediante trasformazioni conformi. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **8** (1950–51), 75–81 (1952).

The author obtains a class of conformal representations of the circle onto a cascade of curved aerofoils, usually without thickness, and discusses the use of these to obtain aerodynamic properties of such cascades.

M. J. Lighthill (Manchester).

Curtis, A. R. Note on the application of Thwaites' numerical method for the design of cambered aerofoils. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2665** (**12,154**), 13 pp. (1952).

Monaghan, R. J. A method of designing corner channels and cascades on a hyperbolic base-line. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2464** (**9522**), 10 pp. (1951).

Incompressible, two-dimensional flows are determined for a channel with given change of direction and given expansion, i.e., ratio of output to input speeds. The channel is mapped onto an auxiliary $\zeta=\xi+i\eta$ plane by the function $z=\cosh\zeta$. An intermediate streamline $\psi=0$ is assumed to coincide with one of the lines $\eta=\eta_0$, which is a hyperbola in the physical z-plane; the channel walls are given by two nearby streamlines. An arbitrary (analytic) speed distribution $f_0(\xi)$ is assumed along this intermediate streamline, subject to the given upstream and downstream speeds. The flow is then completely determined by choosing for the complex velocity (in the ζ-plane) the analytic function $f_0(\zeta-i\eta_0)$. Profiles, and speed and pressure distributions are given for certain choices of $f_0(\xi)$ for a 60° turn and 2:1 expansion, conditions that occur in the design of stator blades for spinner-driven engine cooling fans.

P. W. Ketchum (Urbana, Ill.).

Tetervin, Neal, and Lin, Chia Chiao. A general integral form of the boundary-layer equation for incompressible flow with an application to the calculation of the separation point of turbulent boundary layers. NACA Rep. no. **1046**, ii+19 pp. (1951).

Issued earlier as NACA Tech. Note no. **2158** (1950); these Rev. **12**, 450.

Bouniol, F., et Eichelbrenner, E. A. Calcul de la couche-limite laminaire compressible. Méthode rapide applicable au cas de la plaque plane. Recherche Aéronautique no. 28, 17–20 (1952).

By the transformation due to R. Legendre [Convection de la chaleur en régime permanent, Dunod, Paris, 1949] and Hantzsche and Wendt's substitution [Jahrbuch 1940 der Deutschen Luftfahrtforschung I517–I521; these Rev. 9, 314], the author transforms the boundary layer equations into two total differential equations for two unknown functions of the velocity component parallel to the plate. For arbitrary Prandtl number, the temperature field is shown to be solvable by successive approximations and for each step the equation can be integrated by quadrature. Explicit expressions for temperature are given up to the second approximation. *Y. H. Kuo* (Ithaca, N. Y.).

Quick, August Wilhelm, und Schröder, Kurt. Verhalten der laminaren Grenzschicht bei periodisch schwankendem Druckverlauf. Math. Nachr. 8, 217–238 (1952).

Tatsumi, Tomomasa. Remarks on "stability of the laminar parabolic flow". Physical Rev. (2) 87, 1127–1128 (1952).

Tetervin, Neal, and Levine, David A. A study of the stability of the laminar boundary layer as affected by changes in the boundary-layer thickness in regions of pressure gradient and flow through the surface. NACA Tech. Note no. 2752, 83 pp. (1952).

Coburn, N. The "independent scalars" in homogeneous turbulence. Amer. J. Math. 74, 296–306 (1952).

The author develops a method of generating solenoidal correlation tensors, with applications to isotropic and axial symmetric turbulence. *H. P. Robertson.*

Šapošnikov, I. G. On the theory of weak convection. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 22, 826–828 (1952). (Russian)

An approximate method is outlined for determining the velocity and temperature distribution in a space within a solid body for the case of weak free convection. An imposed constant temperature gradient in the solid is assumed known, providing the single boundary condition. This enters into the Grashof number which is used as the approximating parameter. The solutions are obtained in series form in terms of the Grashof number. The coefficients are given as solutions of linear equations developed from the general free convection equations for the fluid medium. *N. A. Hall.*

Drahlin, E. On heat convection in a spherical cavity. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 22, 829–831 (1952). (Russian)

The method of Šapošnikov [cf. the preceding review] is applied to the case of weak laminar convection in a spherical cavity. Zeroth and first order terms are obtained in explicit form. *N. A. Hall* (Minneapolis, Minn.).

Žuhovickiĭ, E. M. On free steady convection in an infinite horizontal tube. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 22, 832–835 (1952). (Russian)

The method of Šapošnikov [cf. the second preceding review] is applied to the problem of weak laminar convection in a long horizontal tube. The zeroth, first, and second order terms are obtained. Results are computed for one numerical case correlating satisfactorily with reported experimental data. *N. A. Hall* (Minneapolis, Minn.).

Bergman, Stefan. Operatorenmethoden in der Gasdynamik. Z. Angew. Math. Mech. 32, 33–45 (1952).

A comprehensive report (without proof) on the applications of the author's well-known method to two-dimensional gas dynamics. *L. Bers* (New York, N. Y.).

Martin, Monroe H. A new approach to problems in two dimensional flow. Quart. Appl. Math. 8, 137–150 (1950).

This paper is concerned chiefly with two-dimensional steady compressible flow. The pressure and the stream-function are regarded as independent variables and it is assumed that the (numerical value of the) velocity is specified as a function of these variables. Using the lift- and drag-functions of H. Bateman the author derives a system of equations which is equivalent to the equations of motion and of continuity. However, the dependent variables of the new system do not possess an immediate physical meaning. The equivalence of the two systems breaks down only if the stream-lines coincide with the isobars of the flow. Generalising a result of Prim the author shows that in that case the stream-lines are either concentric circles or parallel straight lines (provided the density is a function of the pressure along the stream-lines).

The problem is next reduced to a single Monge-Ampère equation and the classical Cauchy-Kowalewski theorem is used to show that if the fluid velocity is a prescribed function of pressure and stream-function, and if an arc is specified in the plane, then there exists a field of flow for which that arc constitutes a stream-line. Finally, the paper includes a derivation of the substitution principle of Munk and Prim for two-dimensional flow and a brief reference to unsteady flow, which has been considered elsewhere in more detail by the same author.

The usefulness of the present paper is limited by the fact that the information required for its application is not in general available a priori for a concrete problem. However, similar remarks apply to the hodograph method, and in the present case also, physically interesting types of flow might still be discovered by means of the present method by a fortunate choice of the initial data. *A. Robinson.*

***Mathur, Prem N. On the solutions of Chaplygin's equation by means of Kummer's formula.** Proceedings of the Midwestern Conference on Fluid Dynamics, 1950, pp. 99–108. J. W. Edwards, Ann. Arbor, Michigan, 1951.

C'est une étude des équations du mouvement plan d'un gaz parfait dans le domaine subsonique pour un nombre de Mach voisin de un. Les équations sont traitées dans le plan de l'hodographe et la méthode consiste à approcher l'équation d'état par une série de la forme $p=A-\sum_{k=1}^{k=n} B_k/\rho^k$. La résolution du problème se ramène à celle d'une équation différentielle du type de Kummer. Cette théorie permet de calculer l'écoulement de l'espèce considérée autour d'un obstacle, lorsque le mouvement est connu dans le cas d'un fluide incompressible. *R. Gerber* (Grenoble).

de Kármán, Théodore, et Fabri, Jean. Écoulement transsonique à deux dimensions le long d'une paroi ondulée. C. R. Acad. Sci. Paris 231, 1271–1274 (1950).

The speed of perturbations (u, v) with respect to an uniform escape of a moving two-dimensional, irrotational, compressible flow without viscosity can be explored by integrals of Prandtl-Glauert's differential equations

$$(*)\qquad (1-M^2)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\quad \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0,$$

M being Mach's number. The authors discuss first a solution in the case $M<1$. If $M\to 1$, the authors replace the system (*) by the system

$$(^{**})\qquad \frac{\partial^2 x}{\partial \eta^2}+\frac{1}{\xi}\frac{\partial^2 x}{\partial \xi^2}-\frac{1}{\xi^2}\frac{\partial x}{\partial \xi}=0,\quad \frac{\partial^2 y}{\partial \xi^2}+\xi\frac{\partial^2 y}{\partial \eta^2}=0$$

($\xi=(\gamma+1)(u/v^*)$, $\eta=(\gamma+1)(v/v^*)$; v^* critical speed), and discuss the solutions

$$y_1=\left[(\eta+c)^2+\frac{4}{9}\xi^3\right]^{-1/6},\quad y_2=\xi\left[(\eta+c)^2+\frac{4}{9}\xi^3\right]^{-5/6},$$

the combinations $y=y_1+y_2$ and $x=x_1+x_2$ (integrals of the first equation (**)), and the influence of suitable specialized integration constants. *M. Pinl* (Dacca).

Legras, Jean. Remarque sur les ondes de choc en écoulement plan. C. R. Acad. Sci. Paris **234**, 1432–1434 (1952).

The author's principal result on shock waves in symmetrical plane flow past airfoils is that the asymptotic form of the distance between front and rear shocks at a given distance from the line of flight depends only on the Mach number and the maximum thickness of the airfoil, to first order. For some rather more accurate results, see Friedrichs [Communications on Appl. Math. 1, 211–245 (1948); these Rev. 10, 638]. *M. J. Lighthill* (Manchester).

Maslen, Stephen H. Supersonic conical flow. NACA Tech. Note no. **2651**, 32 pp. (1952).

A numerical procedure is evolved, based on the method of characteristics and the relaxation process, for solving the non-linear equations of supersonic conical flow. The process includes the determination of the shock wave shape. It is applied here to the case of a flat plate delta wing in the form of an isosceles right-angled triangle in a uniform stream of Mach number 3 and angle of attack 12°. In particular, the surface pressure distribution is calculated; everywhere the pressure exceeds that given by linear theory, though by varying amounts; the positions of discontinuous pressure gradient are changed.

The reviewer, judging from his analysis of strength of shocks in conical flow to second order [Philos. Mag. (7) **40**, 1202–1223 (1949); these Rev. 11, 625], would suggest that possibly a weak shock really replaces the discontinuity in gradient on the expansion side of the wing and that the numerical procedure may perhaps be not quite sensitive enough to reproduce this. But whether or not this is so, the techniques evolved by the author open up very great possibilities of advance in our knowledge of supersonic flow over wings. *M. J. Lighthill* (Manchester).

Sauer, Robert. Iterationsverfahren zur Berechnung von Unterschallströmungen um Profile und axial angeblasene Drehkörper. Math. Nachr. **8**, 213–216 (1952).

Chen, Yu Why. Supersonic flow through nozzles with rotational symmetry. Comm. Pure Appl. Math. **5**, 57–86 (1952).

The paper is concerned with the following existence problem: suppose that in axisymmetrical steady supersonic flow the flow variables are given on both a forward and a backward characteristic from a point on the axis of symmetry. Then does there exist a continuous flow with continuous velocity gradients in the intervening sector? The question is answered in the affirmative if the prescribed values of the radial velocity on the characteristics tend to zero like $O(r^\gamma)$ as the radius r tends to zero, where $\gamma>\frac{1}{2}$. This excludes the case where discontinuities in velocity gradient occur on the leading characteristic. *M. J. Lighthill* (Manchester).

Van Dyke, Milton D. Practical calculation of second-order supersonic flow past nonlifting bodies of revolution. NACA Tech. Note no. **2744**, 62 pp. (2 plates) (1952).

Ursell, F., and Ward, G. N. On some general theorems in the linearized theory of compressible flow. Quart. J. Mech. Appl. Math. **3**, 326–348 (1950).

The linearized equations of motion for steady potential flow of a compressible fluid past a thin wing with sharp trailing edge are used to treat the problems of aerodynamic forces, uniqueness of solutions and general reversed-flow theorems. It is shown that when there is a vortex wake, the formula for the components of force correct to the second order consists of two parts: a surface integral taken over any surface enclosing the body and a line integral taken along both sides of the trailing edge. The latter line integral, which was previously neglected, involves the velocity potential on both sides of the trailing edge. Therefore, it is concluded that the calculation of the force correct to the second order requires generally the information about discontinuities of the velocity potential to the same order. The leading-edge suction force is not included and must be taken into account when the situation demands. The formula for this force is given. In a similar manner, the moment is also calculated.

By a quadratic identity involving drag, the authors have shown that the linearized problem of a compressible fluid has a unique solution subject to a set of boundary conditions, for both subsonic and supersonic flows. By a slightly different identity, a general reversed-flow theorem is obtained. This theorem is true for both subsonic and supersonic flows and can be specialized to include all flow-reversal theorems obtained by previous authors [von Kármán, J. Aeronaut. Sci. 14, 373–402 (1947); these Rev. 9, 111; Hayes, North Amer. Aviation Rep. no. **AL-222** (1947); Brown, NACA Tech. Note no. **1944** (1949); these Rev. 11, 753; Flax, J. Aeronaut. Sci. 16, 496–504 (1949); these Rev. 11, 224]. The reviewer calls attention to an independent study in which similar general results concerning reverse-flow theorems were obtained by different methods [see the paper reviewed below]. *Y. H. Kuo* (Ithaca, N. Y.).

Flax, A. H. General reverse flow and variational theorems in lifting-surface theory. J. Aeronaut. Sci. **19**, 361–374 (1952).

"By introducing the integral equation and auxiliary conditions adjoint to the integral equation and Kutta condition of lifting-surface theory, certain general theorems in lifting-surface theory are derived. It is shown that the adjoint to the integral equation of a lifting surface is the integral equation for the same surface in reverse flow and that to satisfy the requirements on such an adjoint, the Kutta condition must also be imposed on the solutions of the equation for the surface in reverse flow. The adjoint relationship between wings in direct and reverse flow is shown to lead to a general class of relationships between the characteristics of wings in direct and reverse flow for either subsonic or supersonic speeds. Also, it is shown that the total lift, pitching moment, and rolling moment on a wing of any camber and twist may be expressed as a closed-form integral over certain pressure distributions of the same wing in reverse flow.

These distributions correspond to the case of the wing in reverse flow at constant angle of attack, pitching with uniform velocity, and rolling with uniform velocity. The formulas may be regarded as generalizations of the Munk two-dimensional airfoil integral relations to which they reduce for infinite aspect ratio. Finally, an adjoint variational principle for lifting surfaces in subsonic flow is obtained. In combination with the form of the Rayleigh-Ritz method, this is shown to lead to suitable methods for approximate solution of lifting-surface problems. On the basis of the variational principle, certain apparently arbitrary or intuitive steps in the approximate theories of Prandtl, Weissinger, Reissner, and Lawrence are shown to be logical applications of the same method". (Author's summary.) Cf. the paper reviewed second below. *J. W. Miles.*

Flax, A. H. The reverse-flow theorem for nonstationary flows. J. Aeronaut. Sci. **19**, 352–353 (1952).

For a thin wing oscillating harmonically in a uniform stream, the reverse-flow theorem:

$$\iint \bar{p}(x, y, t)w(x, y, t)ds = \iint p(x, y, t)\bar{w}(x, y, t)ds$$

is proved, where p is the instantaneous value of lifting pressure and w is the instantaneous value of prescribed vertical velocity at any point on a wing plan-form; $\bar{p}$ and $\bar{w}$ are the same quantities for a flow around the same plan-form with the direction of the free-stream velocity reversed and the integrations are over the wing plan-form.

Y. H. Kuo (Ithaca, N. Y.).

Heaslet, Max A., and Spreiter, John R. Reciprocity relations in aerodynamics. NACA Tech. Note no. **2700**, 38 pp. (1952).

Following a discussion of reciprocity theorems in classical physics, general reciprocity theorems for linearized compressible flow past thin wings, previously given in their most general form by Ursell and Ward [cf. the second preceding review], following earlier special cases given by von Kármán, Munk, Hayes, Brown, Harmon, and Flax, are derived by applying Green's theorem to the governing differential equation. Theorems for both steady and unsteady [cf. the paper reviewed above] flow are found with no restrictions on either wing plan form on Mach number beyond those implicit in the linearization. Many interesting and valuable examples are given; e.g., the lift build up on a wing starting impulsively from rest is related to the lift build up on the same wing entering a sharp edged gust in the reverse direction. In its general approach, the work complements a recent paper by Flax [cf. the two preceding reviews], who develops essentially similar results with the aid of the variational principle associated with the governing integral equation. *J. W. Miles* (Los Angeles, Calif.).

Robinson, A., and Hunter-Tod, J. H. Bound and trailing vortices in the linearised theory of supersonic flow, and the downwash in the wake of a delta wing. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2409 (11,296)**, 14 pp. (1952).

Although the concepts of bound and trailing vortices are frequently discarded in treating supersonic lifting-surface problems, they may be useful for calculating the flow field when the lift distribution is known. For example, the downwash directly behind a flat delta wing at incidence, with subsonic leading edge, is calculated here. The formulas applied are from an earlier paper by Robinson [Coll. Aeronaut. Cranfield. Rep. no. **9** (1947); these Rev. **10**, 74], where certain "hyperbolic" vector operators and Hadamard integration were employed. The downwash angle increases from the trailing edge aft, very nearly reaching its asymptotic value within two chord lengths. The ratio of downwash angle to angle of incidence, which is always less than one, decreases with increasing Mach number and apex angle.

W. R. Sears (Ithaca, N. Y.).

Robinson, A. Aerofoil theory for swallow tail wings of small aspect ratio. Aeronaut. Quart. **4**, 69–82 (1952).

Modified version of Coll. Aeronaut. Cranfield Rep. no. **41** (1950); these Rev. **12**, 452.

Frazer, R. A. Possio's subsonic derivative theory and its application to flexural-torsional wing flutter. I. Possio's derivative theory for an infinite aerofoil moving at subsonic speeds. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2553 (4932, 5916)**, 1–16 (1951).

The paper reviews and amplifies the linearized derivative theory of Possio for an oscillating thin airfoil at subsonic speed. Essentially with the concept of acceleration potential, the elementary solutions of source and doublet are constructed and superimposed. The intensity of doublet distribution over the chord is represented by a series and the series coefficients are determined from the normal velocity at an appropriate number of points. The derivative values are calculated for a Mach number of 0.7 and for the values of the frequency parameter λ from 0 to 5.0. It is found necessary that satisfactory accuracy can be obtained with more points located on the airfoil as the value of λ increases.

C. C. Chang (Manchester).

Frazer, R. A., and Skan, Sylvia W. Possio's subsonic derivative theory and its application to flexural-torsional wing flutter. II. Influence of compressibility on the flexural-torsional flutter of a tapered cantilever wing moving at subsonic speed. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2553 (4932, 5916)**, 17–22 (1951).

The paper gives the calculations based on Possio's subsonic derivative theory [Part I; see the preceding review] and on vortex strip theory in order to determine the influence of compressibility and altitude on the critical speed for flexural torsional flutter of a tapered cantilever wing. It is found that for normal stiffness ratio and a given altitude, the compressibility effect on critical speed for flutter is insignificant at Mach number 0.7. The critical speed at 30,000 feet altitude is about 5–8% lower than that at sea level. The corresponding divergence speed ratio confirms a result of Theodorsen and Garrick. *C. C. Chang.*

Miles, John W. On the general solution for unsteady motion of a rectangular wing in supersonic flow. J. Aeronaut. Sci. **19**, 421–422 (1952).

The author claims to obtain the general solution of a rectangular wing moving unsteadily in a supersonic flow when he carries out a Fourier transform in time t and a Laplace transform in longitudinal coordinate x and compares with some known particular results. An alternative approach to the general solution is also proposed and briefly indicated. It is remarked that the general solution can be extended to any wing having a supersonic leading edge of

any shape and straight subsonic side edges (not trailing) as long as there is no influence between the latter.
C. C. Chang (Manchester).

Costello, George R., Cummings, Robert L., and Sinnette, John T., Jr. Detailed computational procedure for design of cascade blades with prescribed velocity distributions in compressible potential flows. NACA Rep. no. **1060**, 14 pp. (1952).

Bryson, A. E. Note on aerodynamic heating with a variable surface temperature. Quart. Appl. Math. **10**, 273–275 (1952).

This note deals with the problem of unsteady aerodynamic heating by the flow of an incompressible viscous fluid started impulsively from rest over an infinite flat plate. The energy equation for the temperature distribution for this case will be linear, inhomogeneous, with constant coefficients, and is solved with prescribed initial conditions and rate of heat flow through the plate. The results are carried over to the case of steady boundary-layer flow over a semi-infinite plate by means of the familiar approximation. Upon comparison with Lighthill's more exact solution [Proc. Roy. Soc. London. Ser. A. **202**, 359–377 (1950); these Rev. **12**, 218] for this case, it is found that the Reynold's number dependence, at least, is correct.
Y. H. Kuo.

Kuo, Hsiao-Lan. Three-dimensional disturbances in a baroclinic zonal current. J. Meteorol. **9**, 260–278 (1952).

The equations of motion and of continuity are expressed in terms of a coordinate system in which the potential temperature replaces the linear vertical coordinate. Adiabatic conditions must, therefore, be assumed and the surfaces of constant potential temperature must not intersect each other. The equations governing the perturbations of a zonal baroclinic current, are set up both for non-viscous and for viscous flow. The basic current is assumed to have a velocity that increases linearly with height but is independent of latitude. A careful analysis of the nature of the damped and amplified types of oscillatory perturbations of this current is made and it is shown that most disturbances are unstable. The amplitude of the disturbances of greatest instability may be doubled in 24 hours. Disturbances of this kind produce a downward transport of zonal momentum and a northward transport of heat. The perturbation kinetic energy is produced by the horizontal pressure-force at low levels in the atmosphere. *G. C. McVittie* (Urbana, Ill.).

McVittie, G. C. Theory of development and of thickness patterns. Tellus **4**, 8–20 (1952).

The author's objective is to find the place of recent work on the atmospheric circulation by Rossby, Sutcliffe, and Charney within the framework of the hydrodynamics of perfect non-conducting fluids. To this end he first transforms the fundamental equations, including the vorticity equation, to coordinates x, y, p, where p is the pressure and where x, y are rectangular coordinates in a plane tangent to the earth at a specific position. Variation of Coriolis acceleration with latitude is neglected. To justify subsequent approximations obtained by dropping certain terms in the equations, the author employs observed numerical values of the quantities occurring. The usual vertical hydrostatic equation is obtained by supposing the isobaric surfaces to be nearly horizontal. Simple definitions of geostrophic and ageostrophic winds are given. The author formulates specific mathematical assumptions which are sufficient to derive the equations used by Rossby, Sutcliffe, and Charney. He emphasizes the fact that these equations thus appear as defining specific models of circulation. Whether or not the hypotheses are valid, with the consequence of the appropriateness or inappropriateness of their results for describing the actual circulation of the atmosphere, is then an experimental matter. To those who would like to understand in hydrodynamical terms the current research trends of dynamical meteorology, this paper should be quite helpful.
C. Truesdell (Bloomington, Ind.).

van Mieghem, J. Some remarks on the angular momentum balance in the atmosphere. Tellus **4**, 135–138 (1952).

The author writes down in spherical coordinates the equations satisfied by the angular momentum of a viscous fluid. He averages these equations along a circle of latitude and discusses the interpretation and magnitude of the terms occurring. *C. Truesdell* (Bloomington, Ind.).

Nedospasov, A. V. On the theory of sound from rotation. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **22**, 579–584 (1952). (Russian)

L'autore studia il suono generato da una sfera rigida ruotante attorno ad un asse fisso, che non coincide con un diametro, in un fluido ideale indefinito. Si suppone che il moto ammetta potenziale di velocità. Per rendere il problema trattabile, il campo viene diviso in due parti, mediante una superficie che racchiude la sfera ruotante. All'interno il fluido viene supposto incompressibile; la soluzione trovata in questa ipotesi serve a dare le condizioni sulla superficie, alle quali deve soddisfare il moto all'esterno; quest'ultimo viene supposto lento, in modo da poter trascurare i termini quadratici nelle equazioni indefinite.
G. Toraldo di Francia (Firenze).

Elasticity, Plasticity

***Lohr, Erwin. Mechanik der Festkörper.** Walter de Gruyter & Co., Berlin, 1952. viii+483 pp. DM 39.60.

As the publisher states in a postscriptum to the preface, the author died before he could complete his three volume textbook of physics of which the present book was to be the first volume. This explains the selection of topics which would otherwise be hard to understand. About the first fifth of the book is devoted to the fundamentals of mechanics, and the second fifth to the statics and dynamics of rigid bodies. Thus, only little more than one half of the volume is concerned with the topics which the title is likely to suggest to most readers, namely, the mechanics of deformable solids. This portion of the book is devoted primarily to the fundamental and general principles of elastostatics and elastokinetics, with an introduction to the theory of plasticity. Vector and tensor methods are used throughout the volume. It is interesting to note how the Gibbs notation which is quite adequate in mechanics of rigid bodies becomes exceedingly cumbersome in mechanics of deformable bodies. *W. Prager* (Providence, R. I.).

Aržanyh, I. S. The fundamental integral equations of the dynamics of an elastic body. Doklady Akad. Nauk SSSR (N.S.) **81**, 513–516 (1951). (Russian)

Let $Q+S$, where S is a sufficiently smooth surface, and Q is either its interior or its exterior, be the domain occupied

by the elastic body. The displacement vector $v(q, t)$, where q denotes a point of the domain, satisfies in Q the equation

$$\frac{\partial^2 v}{\partial t^2} = R + \alpha \operatorname{grad} \operatorname{div} v - \beta \operatorname{rot} \operatorname{rot} v,$$

where α and β are elastic constants and R is a known vector, satisfies the initial conditions $v(q, 0) = v_0$, $\partial v(q, 0)/\partial t = v_0'$, and one of the boundary conditions: (a) first boundary-value problem, $v(s, t) = v_s$ on S; (b) second boundary-value problem,

$$2\mu \frac{\partial v}{\partial n} + \lambda n \operatorname{div} v + \mu n \times \operatorname{rot} v - h(s) v(s, t) = P_n(s, t)$$

on S. In both boundary-value problems, the author obtains integral equations which are satisfied by the Laplace transforms of div v and rot v.

J. B. Dias.

*van Langendonck, Telemaco. **Funções ortogonais na resulução de problemas da teoria da elasticidade. Tomo I. Generalidades e torção. [Orthogonal functions in the solution of problems of the theory of elasticity. Vol. I. Generalities and torsion.]** Associação Brasileira de Cimento Portland, São Paulo, 1952. viii+69 pp.

The method employed by the author in connection with the two-dimensional Dirichlet problem $\nabla^2 u = 0$ inside C, $u = f$ on C, would consist in first determining a set of n harmonic polynomials $\phi_k(x, y)$ such that $\int_C \phi_i \phi_j ds = \delta_{ij}$. The function $\sum_{k=1}^{n} (\int_C \phi_k f ds) \phi_k(x, y)$ would constitute the approximate solution. Appropriate modifications of this procedure are used to obtain detailed approximate solutions of the Saint Venant torsion problem for cylindrical bars of various shapes. No error estimates are provided.

F. B. Hildebrand (Cambridge, Mass.).

Prokopov, V. K. On a plane problem of the theory of elasticity for a rectangular region. Akad. Nauk SSSR. Prikl. Mat. Meh. **16**, 45–56 (1952). (Russian)

The author had previously [same journal **14**, 527–536 (1950); these Rev. **13**, 88] applied the procedure employed by Papkovič [C. R. (Doklady) Acad. Sci. URSS **27**, 334–338 (1940); these Rev. **2**, 232] and Lur'e [Akad. Nauk SSSR. Prikl. Mat. Meh. **6**, 151–168 (1942); these Rev. **5**, 138] in plane elasticity problems, in order to find the bending of a circular plate under axially symmetric loads. In the present paper the author extends the procedure of Papkovič and Lur'e, applying it to the problem of determining the state of stress in a bar of rectangular cross section $-a \leqq x \leqq a$, $-b \leqq y \leqq b$, when the only force acting on $y = b$ is a concentrated load (in the $-y$ direction) at the point $x = c$, $y = b$; $y = -b$ is free of load; and the sides $x = \pm a$ are held fixed (the actual conditions required are that $u = 0$ on $-b < y < b$, $v = 0$ for $y = 0$, on $x = \pm a$, where u and v are the displacements in the x and y directions respectively). The corresponding biharmonic stress function and the displacements are obtained explicitly. The case of distributed loads along $y = \pm b$ and the limiting case of b/a small are discussed subsequently.

J. B. Dias (College Park, Md.).

Slade, J. J., Jr. The elastic axes of a one-mass elastically supported system. Quart. Appl. Math. **10**, 278–280 (1952).

Aquaro, Giovanni. Sopra un teorema di media per le equazioni dell'Elasticità. Rend. Sem. Fac. Sci. Univ. Cagliari **21** (1951), 43–46 (1952).

The author obtains a short and direct demonstration of his theorem of mean value [Rivista Mat. Univ. Parma **1**, 419–424 (1950); these Rev. **12**, 770].

C. Truesdell.

Milne-Thomson, L. M. Finite deformations and elasticity. Revista Mat. Hisp.-Amer. (4) **12**, 9–35 (1952). (Spanish)

Essentially an expansion of the author's earlier paper [Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 33–39; these Rev. **12**, 63], which in turn was essentially a translation into dyadic notation of Murnaghan's presentation [Amer. J. Math. **59**, 235–260 (1937)] of the classical theory of finite elastic strain.

C. Truesdell.

Adkins, J. E., and Rivlin, R. S. Large elastic deformations of isotropic materials. IX. The deformation of thin shells. Philos. Trans. Roy. Soc. London. Ser. A. **244**, 505–531 (1952).

Contrary to the implication of the title, this paper deals only with two special problems. (1) A plane sheet is symmetrically extended, then clamped on a circular boundary, then subjected to uniform internal pressure. (2) A spherical shell is subjected to uniform internal pressure. The treatment of problem (2) is very brief and complements the work of Green and Shield [Proc. Roy. Soc. London. Ser. A. **202**, 407–419 (1950); these Rev. **12**, 218].

The authors' treatment of problem (1) begins with the usual equilibrium equations of shell theory, which, provided they be understood to apply in the deformed condition of the body, are valid for strains of any magnitude [cf. the reviewer, Trans. Amer. Math. Soc. **58**, 96–166 (1945), see §8; Bull. Amer. Math. Soc. **54**, 994–1008 (1948), see §8; these Rev. **7**, 231; **10**, 341]. The authors put all shear and moment resultants equal to zero. The usual equations of the extensional or "membrane" theory then reduce to $\partial(\rho T_1)/\partial\rho = T_2$, $\kappa_1 T_1 + \kappa_2 T_2 = P$, where T_i are the tension resultants, κ_i the principal curvatures, and ρ the distance from the axis of symmetry. The difference from the usual problem in small deflection theory is that ρ and κ_i are unknown functions of the coordinates in the undeformed sheet. The authors next set the principal stress $t_3 = 0$, despite the fact that the true boundary condition is $t_3 = -P$ on the inner surface, stating this procedure to be sufficiently accurate "since P is very small compared with T_1 and T_2" [note: should read "t_1 and t_2"]. Then they set $T_i = \lambda_3 h t_i$, $i = 1, 2$, where λ_3 is the normal extension and h the thickness. This assumption enables them to bring in the deformations through the usual formulae of finite strain theory for the t_i in terms of the strain energy. Their final formulation is a very elaborate system of four first order differential equations for the T_i and κ_i. [The reviewer does not consider the derivation valid. It is possible, however, that more scrupulous treatment of the stress averages, as distinct from the stresses, might show the authors' equations to be correct as a first order approximation.] The remainder of the treatment of problem (2) consists in discussions of the signs of various derivatives at the pole and at the equator, and in numerical solution for certain special strain energies. Comparison with experimental data shows fairly good agreement.

C. Truesdell.

Weinstein, Alexander. On cracks and dislocations in shafts under torsion. Quart. Appl. Math. **10**, 77–81 (1952).

The present paper gives new application of the theory of W. Arndt [Thesis, Göttingen, 1916], extended by the au-

thor [Quart. Appl. Math. 5, 429–444 (1948); Proc. Seventh Internat. Congress Appl. Mech. 1948, v. 1, pp. 108–119; these Rev. 10, 116; 12, 559], van Tuyl [Quart. Appl. Math. 7, 399–409 (1950); these Rev. 11, 474], and Sadowsky and Sternberg [ibid. 8, 113–126 (1950); these Rev. 12, 259] to problems of cracks and dislocations in a shaft under torsion. The theory of shafts of revolution under torsion has been reduced by W. Arndt to the investigation of axially symmetric motion of a fictitious incompressible fluid in a space of five dimensions. A new and simple correspondence principle can be used connecting the method of sources and sinks with electrostatic problems in generalized axially symmetric potential theory. This theory deals with equations of the form

$$y^p\varphi_x(x, y)=\psi_y(x, y),\quad y^p\varphi_y(x, y)=-\psi_x(x, y),\quad p=0, 1, 2, \cdots.$$

The case $p=0$ corresponds to harmonic functions in the plane; for $p=1, 2, 3, \cdots$ the function φ represents an axially symmetric potential in the meridian plane of a space of $p+2$ dimensions. An important identity is $\psi\{p\}=Cy^{p+1}\varphi\{p+2\}$, C = arbitrary constant. This identity permits obtaining from a stream function $\psi\{p\}$ a certain potential $\varphi\{p+2\}$ and vice versa. In the case of shafts, solutions are obtained by putting, for a constant U,

$$\Phi\{p\}=Ux-\varphi\{p\},\quad \Psi\{p\}=U(p+1)^{-1}y^{p+1}-\psi\{p\}.$$

In the usual cases φ and ψ are due to sources distributed in a finite domain. The main problem is to determine the lines $\Psi\{p\}$ = const., the line $\Psi\{p\}=0$ being the line enclosing the singularities. The problem for $\Psi\{p\}$ can be reduced by a correspondence between the closed streamline $\Psi\{p\}=0$ and the level line $\varphi^*=\varphi\{p+2\}=1$ to an electrostatic problem for $\varphi\{p+2\}$. A flat interior circular crack perpendicular to the axis of symmetry of the shaft is defined as a circular disc separating the material without stresses acting across the surface of the disc. The boundary condition of the crack problem is given by $\Psi\{3\}=0$ on both sides of the segment $x=0$, $0\leqq y\leqq b$ in the upper half of the meridian plane, Ψ being an even function of x. By the correspondence principle this problem is reduced to the determination of the electrostatic potential $\varphi\{5\}$ of a disc of radius b in a space of seven dimensions. The author solves this electrostatic problem in ellipsoidal coordinates. According to the classical theory of Volterra a surface of dislocation is a surface of discontinuity for the displacements. In the case of a shaft under torsion the same circular disc as in the crack-problem can be used as the surface of the dislocation. The problem of dislocations is equivalent with the case $\varphi=\varphi\{3\}$ of a potential of a uniform magnetic shell in a five-dimensional space. The solution depends on Bessel functions. *M. Pinl.*

Baldacci, Riccardo F. Un metodo variazionale nel problema della lastra. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 85, 127–137 (1951).

The paper deals with the problem of solving the plate equation (*) $\Delta\Delta w=p/D$ in a plane domain on the boundary S of which the values of w and $\partial w/\partial n$ are assigned, this problem being equivalent to that of minimising the energy integral

$$E(w)=\tfrac{1}{2}D\int\int\left[(\Delta w)^2-2(1-\nu)(w_{xx}w_{yy}-w^2_{xy})-\frac{2pw}{D}\right]dxdy.$$

The main result of the paper is that $E(w)\geqq E(\psi)$ where w is the solution and ψ is arbitrary except that it satisfies (*) and also $I(\psi)=0$ where

$$I(\psi)=\int_S\{\Delta\psi\cdot(w-\psi)_n-(w-\psi)(\Delta\psi)_n$$
$$-(1-\nu)[\psi_x(w_y-\psi_y)_s-\psi_y(w_x-\psi_x)_s]\}ds,$$

the subscripts n, s indicating $\partial/\partial n$, $\partial/\partial s$. The terms in $I(\psi)$ involving w are of course known from the boundary conditions. The author proposes to obtain approximate solutions of (*) by using in $I(\psi)=0$ a sum $\sum_k c_k f_k$ where the f_k satisfy (*). However such a sum will not satisfy (*) unless $\sum_k c_k=1$, and the argument at this point becomes obscure to the reviewer, perhaps owing to misprints. *J. L. Synge.*

Stippes, M., and Hausrath, A. H. Large deflections of circular plates. J. Appl. Mech. 19, 287–292 (1952).

A solution is given of von Kármán's large-deflection plate equations for a simply-supported circular plate under a uniformly distributed load. A perturbation procedure is used to obtain the solution. The method is shown to be valid provided the load is sufficiently small. Numerical results are presented for deflections of somewhat over 0.6 times the plate thickness. The author states that the solution appears as a slowly convergent series [in the reviewer's opinion, this is typical of large deflection analyses]. An advantage of the method is that it can be used directly to compute deflections for concentrated loads. *S. Levy.*

Fletcher, H. J., and Thorne, C. J. Thin rectangular plates on elastic foundation. J. Appl. Mech. 19, 361–368 (1952).

The sine transform method is used to obtain a general solution for the deflections and moments in a thin rectangular plate on an elastic foundation when carrying transverse loads which are continuous in one direction and sectionally continuous in the other. Twenty-four graphs are given presenting numerical results for a variety of loading conditions on a square plate with two opposite edges simply supported and the other two edges clamped, simply supported, or free. In most cases considered, relative stiffnesses of foundation and plate are such that the plate carries the largest portion of load. The authors also give the equation for the case where the load varies sinusoidally with time and show that this equation differs from the steady load equation only in a change in the effective foundation modulus. *S. Levy.*

Šeremet'ev, M. P. Elastic equilibrium of an infinite plate with an inlaid absolutely rigid or elastic disc. Akad. Nauk SSSR. Prikl. Mat. Meh. 16, 437–448 (1952). (Russian)

van Wijngaarden, A. Large deflections of semi-oval rings. Nationaal Luchtvaartlaboratorium, Amsterdam. Report S.313, 7 pp. (1951).

The author first formulates the problem of determining the deflection of a thin elastic rod, of variable flexural rigidity, which is initially arbitrarily curved in a principal plane and subsequently loaded at a number of discrete points by forces and couples in that plane. After outlining a method for solving the problem in terms of infinite series, attention is restricted to the classical case of a rod with uniform rigidity and uniform initial curvature, for which the solution is expressible in terms of elliptic integrals. A detailed study of the solution is followed by its application to the analysis of a semi-oval ring clamped at its ends and subjected to a concentrated transverse load at its midpoint. *F. B. Hildebrand* (Cambridge, Mass.).

Stevens, G. W. H. The stability of a compressed elastic ring and of a flexible heavy structure spread by a system of elastic rings. Quart. J. Mech. Appl. Math. 5, 221–236 (1952).

The bending stability of a thin elastic ring of uniform section, compressed by a uniform radial pressure, is investigated by a method which is appropriate to the consideration of the stability of a cone of heavy flexible material spread by a system of elastic hoops. The analysis is similar to that of Biezeno and Koch [Nederl. Akad. Wetensch., Proc. 48, 447–468 (1945); these Rev. 8, 360], but is somewhat simplified. *F. B. Hildebrand* (Cambridge, Mass.).

Wittrick, W. H. Correlation between some stability problems for orthotropic and isotropic plates under bi-axial and uni-axial direct stress. Aeronaut. Quart. 4, 83–92 (1952).

The following four buckling problems are considered: (a) a rectangular orthotropic plate, with all edges simply supported, subjected to compression on its ends and a known compression or tension on its sides; (b) the same as (a) but with the ends clamped and sides simply supported; (c) a rectangular orthotropic plate, with the ends simply supported and sides clamped, subjected to compression on its ends; and (d) the same as (c) but with all edges clamped. In each case it is shown that the variables involved can all be combined into such a non-dimensional form that a single curve serves to give the value of the end compression required to cause buckling. This curve is identical with the curve of buckling stress coefficient against side ratio for a corresponding isotropic plate under uni-axial compression. In cases (a) and (b) the formulas for the combination of the variables in non-dimensional form follow from the analysis. In cases (c) and (d) they contain an empirically determined factor. The lateral compression is restricted to lie below a certain limit in cases (a) and (b). *H. W. March.*

Melyahoveckiĭ, A. S. The integral equation of the free vibrations of a curvilinear bar. Doklady Akad. Nauk SSSR (N.S.) 85, 513–516 (1952). (Russian)

The author considers the extensional vibrations of a thin bar whose axis is a smooth plane curve, with its ends clamped elastically. The sixth-order differential equation for the tangential displacement together with boundary conditions is replaced by an integral equation, whose eigen-functions have the usual Fourier properties. *F. V. Atkinson.*

Šatašvili, S. H. On steady vibrations with given external forces on the surface of an elastic body. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 615–617 (1951). (Russian)

The problem mentioned in the title has been solved by V. D. Kupradze [Boundary value problems in the theory of vibrations and integral equations, Moscow-Leningrad, 1950], and later by A. M. Kyskov [Doklady Akad. Nauk SSSR 70, 197–200 (1950); these Rev. 11, 702] who obtained an equivalent system of singular integral equations. The present author, employing a method closely related to that of D. I. Šerman [Akad. Nauk SSSR. Prikl. Mat. Meh. 10, 617–622 (1946); these Rev. 8, 361], reduces the boundary value problem in question to the solution of a Fredholm system of integral equations. By employing results of D. I. Šerman [ibid. 11, 259–266 (1947); these Rev. 9, 121] and J. D. Tamarkin [Ann. of Math. 28, 127–152 (1927)], it is shown that this system of integral equations has a solution for "almost all" values of the frequency parameter ω. ■
J. B. Diaz (College Park, Md.).

Sehniašvili, Ė. A. On the determination of the natural modes of vibration of systems with an infinitely large number of degrees of freedom. Soobščeniya Akad. Nauk Gruzin. SSR. 12, 149–154 (1951). (Russian)

The author remarks that a practical method of finding an arbitrary mode of vibration for a system with infinitely many degrees of freedom does not exist up to now, and he claims to fill this gap for an elastic bar of arbitrary form. By Galerkin's method, the problem is reduced to the solution of a system of infinitely many linear homogeneous equations with infinitely many unknowns. The natural frequencies are found by equating the determinant of this system to zero. The remark that at the zeros of this determinant its rank is one is not clear to the reviewer. Finally, the author explains the well-known fact that the vibration of the bar for given initial deflection and velocities can be expressed by the characteristic solutions. *W. H. Muller.*

Maue, A.-W. Die Kantenbedingung in der Beugungstheorie elastischer Wellen. Z. Naturforschung 7a, 387–389 (1952).

In the theory of the diffraction of acoustic or electromagnetic waves at a screen, the solution is determined by the boundary conditions on the faces of the screen, the radiation condition at infinity, and the conditions at the edge of the screen. A similar sort of problem relates to the diffraction of elastic waves, in an elastic solid of infinite extent, where the role of the screen is played by a fracture surface whose two faces are regarded as free boundaries. In the present paper, the nature of the edge conditions is derived by considering the case when the "screen" is a half-plane and the wave-length is so large that the problem is virtually an elastostatic one. This sort of argument tells us what form the edge-conditions should take; a rigorous proof would doubtless be very difficult. *E. T. Copson.*

Onat, Turan. Pekleşen malzemeden mamul prizmatik çubukların burulması. [Torsion of prismatic rods of work-hardening material.] Thesis, Istanbul Technical University, Kutulmus Basımevi, Istanbul, 1951. 48 pp. (7 plates)

The first part of the paper is devoted to the discussion of general stress-strain laws of finite and incremental types and to the derivation of the resulting differential equations for the stress function in the torsion problem. Adopting a finite stress-strain law, the author then discusses the integration of the equation $(f\psi_x)_x+(f\psi_y)_y=-1$ where the stress function $\psi(x, y)$ vanishes on the contour of the simply connected cross section and f is a known monotonically increasing function of $|\mathrm{grad}\,\psi|$. An iterative procedure is developed requiring, at each step, the solution of a Poisson equation the right-hand side of which depends on the permanent strains obtained in the previous step. Numerical results based on the stress-strain diagram of 24ST aluminum alloy are presented for square and rectangular sections.
W. Prager (Providence, R. I.).

Mii, Hisao. Some notes on the plastic deformation of hollow spheres with large strain. II. Theory of unloading. J. Jap. Soc. Appl. Mech. 5, 32–37 (1952).

Continuation of part I [same J. 3, 133–139 (1950); these Rev. 12, 882].

Mandel, Jean. Sur la réactivité des solides. C. R. Acad. Sci. Paris **233**, 1003–1005 (1951).

A delayed elastic Voigt type material is considered, with stress-strain relation $t_{ij}=(\lambda\theta+\lambda'\dot{\theta})\delta_{ij}+2\mu g_{ij}+2\mu'\dot{g}_{ij}$, where t_{ij} is the stress, g_{ij} the strain, θ the dilatation, δ_{ij} the Kronecker delta, λ, λ', μ and μ' constants, and a dot indicates a time derivative. The equations of equilibrium are written in terms of displacements. For constant force and displacement boundary conditions, solutions are shown to be composed of the superposition of a constant distribution satisfying the boundary conditions, and a series of terms with time dependence $e^{-\alpha t}$ and homogeneous boundary conditions. The spectrum of values α, and expansion of initial conditions in terms of the corresponding eigenfunctions are discussed. The influence on the solution of the delay times $\tau=1/\alpha$ is mentioned and the satisfaction of variable boundary conditions by superposition of incremental solutions.

E. H. Lee (Providence, R. I.).

Wang, Ming Chen, and Guth, Eugene. Statistical theory of networks of non-Gaussian flexible chains. J. Chem. Phys. **20**, 1144–1157 (1952).

The authors begin with a critical evaluation of the statistical theories of a rubber-like material, regarded as an assembly of long chain molecules. Their aim is to obtain a statistical theory which agrees with experimental data on the stretching of rubber and with the predictions of the theory of finite elastic strain. They regard the treatment in the present paper as more complete and accurate than any heretofore published, that of Isihara, Hashitsume, and Tatibana included [same J. **19**, 1508–1512 (1951); these Rev. **13**, 1004]. The authors adopt Rayleigh's exact distribution for a single perfectly flexible chain (random walk problem), but replace his series expansion by another, valid for a chain of many links even when the length is of the same order as the number. Hence they calculate the distribution function for a network of chains, first for the case when the free junctions are not restricted and for the case when these are at their most probable positions. From their analysis they conclude that the results for the latter case, which are simpler, approximate sufficiently those for the former. Therefore they thenceforth treat all junctions as fixed. Hence they are able to discuss the effect of homogeneous extensions in three mutually perpendicular directions. By retaining only the first two terms in their series, they are able to get a strain energy containing terms of degrees 1, 2, and 3 in the extensions. All coefficients are proportional to the absolute temperature. The result differs from that of the continuum theory, whose first approximation for large strain of incompressible materials was worked out by Mooney [J. Appl. Phys. **11**, 582–592 (1940)] in that the coefficients are not arbitrary constants, but are related to one another.

The authors use statistical methods only to obtain a form for the strain energy. Thence to derive the stresses they use the stress-strain relations of the classical theory of finite strain in the form given by Rivlin [Philos. Trans. Roy. Soc. London. Ser. A. **241**, 379–397 (1948); these Rev. **10**, 340]. They conclude that their results agree with experimental data on the extension of rubber qualitatively, but not quantitatively. They state that in order to obtain a more satisfactory statistical theory it would be necessary to take into account inter- and intramolecular forces, which are entirely neglected in the present treatment.

C. Truesdell.

Viswanathan, K. S. On the characteristic vibrations of linear lattices. Proc. Indian Acad. Sci., Sect. A. **35**, 265–276 (1952).

Die Ausbreitung von elastischen Wellen in einem linearen Gitter, in dem jedoch in jeder Elementarzelle p verschiedene Partikel enthalten sind, wird untersucht. Angenommen wird dabei, dass jedes Partikel nur mit seinen unmittelbaren zwei Nachbaren in Wechselwirkung steht. Die daraus folgenden bekannten Differentialgleichungen werden mit Hilfe des Ansatzes

$$x_{q,r}=f_q e^{i(\omega t-r\theta)} \tag{1}$$

gelöst, wo sich r auf die Zellen und q auf die Partikeln innerhalb einer Zelle bezieht. Durch Einsetzen von (1) in die Differentialgleichungen und Elimination der f_q erhält man die bekannte Determinantengleichung für ω^2; aus derer Besprechung folgt dann, dass wenn $\theta=\pi$ ist, die Gruppengeschwindigkeiten von allen p Frequenzen verschwinden, ist dagegen $\theta=0$, so wird eine Frequenz gleich Null und die dazugehörende Gruppengeschwindigkeit verschwindet nicht, für die übrigen $p-1$ ist das jedoch der Fall. Also gibt es $2p-1$ charakteristische Frequenzen für die die dazugehörenden Gruppengeschwindigkeiten verschwinden. Weiter wird gezeigt, dass eine lokalisierte Störung im Gitter asymptotisch in die Superposition der $2p-1$ charakteristischen Schwingungen des Gitters übergeht. Alle diese Resultate gehen für $p=1$ in die von Hamilton [Mathematical papers, v. 2, Cambridge, 1940, p. 452] und für $p=2$ in die von Nagendra Nath und S. K. Roy [dieselben Proc. **28**, 289–295 (1948); diese Rev. **10**, 489] über. Der dreidimensionale Fall wurde von C. V. Raman [ibid. **18**, 237–250 (1943)] behandelt.

T. Neugebauer (Budapest).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Wolf, E. On a new aberration function of optical instruments. J. Opt. Soc. Amer. **42**, 547–552 (1952).

A new characteristic function is introduced with the help of a so-called Gaussian reference sphere. This sphere is defined for a given ray so as to have its center at the Gaussian image point and to pass through the center of the exit pupil. The new characteristic function is simply the optical path from the object point to the intersection of the ray with the Gaussian reference sphere. By considering a wave surface passing through the center of the exit pupil in comparison with the Gaussian reference sphere, one can readily determine how the wave surface differs from spherical form. The aberration characteristic can be expressed in terms of three symmetric variables and expanded in a series, which shows the close connection with the well-known angle characteristic and the Seidel eikonal S of Schwarzschild as well as the five Seidel coefficients of primary spherical aberration, coma, astigmatism, curvature and field distortion. The new function appears to be well-suited for investigations of aberration effects on the basis of geometrical optics or of diffraction theory.

E. W. Marchand (Rochester, N. Y.).

Herzberger, M. The contributions of the single surfaces to the diapoint coordinates. J. Opt. Soc. Amer. **42**, 544–546 (1952).

Let x, y, z be the coordinates of an object point and x', y', z' those of its diapoint. Let ξ, η, φ; ξ_ν, η_ν, φ_ν; and ξ', η', φ' be the optical direction cosines of an object ray and the corresponding intermediate and image rays respectively. Then

$$\frac{1}{x'\xi'}=\frac{1}{x\xi}+\sum\frac{\phi_\nu}{\xi_\nu\xi_\nu'},$$

$$\frac{1}{y'\eta'}=\frac{1}{y\eta}+\sum\frac{\phi_\nu}{\eta_\nu\eta_\nu'},$$

where ϕ_ν is the power of the νth surface. Also $1/z'\varphi'$ is given as a continued fraction containing $\varphi_\nu/\varphi_\nu\varphi_\nu'$ and the center distances c_ν multiplied by φ_ν. These formulae contain as a special case the so-called Petzval formula and in the case of meridional rays reduce to a formula of Kerber. Furthermore, they can evidently be used to predict the effect of individual surfaces changes on the quality of the image.

E. W. Marchand (Rochester, N. Y.).

Herzberger, M., and Marchand, E. Image error theory for finite aperture and field. J. Opt. Soc. Amer. **42**, 306–321 (1952).

Questa ricerca si riallaccia ai lavori di Herzberger [stesso J. **26**, 197–204 (1936); **38**, 736–738 (1948); questi Rev. **10**, 220] sulla teoria dei diapunti. Per un'apertura e un campo finiti le aberrazioni sono rappresentate da due funzioni B, C; se è nulla C, l'immagine è simmetrica; se è nulla anche B, l'immagine e stigmatica. Le funzioni B e C si lasciano esprimere per mezzo delle derivate seconde della caratteristica mista V rispetto a tre opportune variabili. Gli autori riescono poi ad esprimere queste derivate di V per mezzo delle derivate della caratteristica angolare W, ciò che permette di calcolare facilmente l'effetto di uno spostamento del piano oggetto sulle aberrazioni. La correttezza e l'utilità delle formule viene verificata su alcuni esempi particolarmente semplici.

G. Toraldo di Francia (Firenze).

Weyl, Hermann. Kapazität von Strahlungsfeldern. Math. Z. **55**, 187–198 (1952).

This paper is a more detailed version of that of the author in Proc. Nat. Acad. Sci. U. S. A. **37**, 832–836 (1951); these Rev. **13**, 802. In addition to the complete proof of existence for the exterior problem through the use of the concept of "radiation capacity", there is an additional section devoted to the structure of the "capacity matrix."

W. K. Saunders (Washington, D. C.).

Senior, T. B. A. Diffraction by a semi-infinite metallic sheet. Proc. Roy. Soc. London. Ser. A. **213**, 436–458 (1952).

The problem treated is the two-dimensional one of the reflection of a plane wave by the semi-infinite plate $x>0$. The plate is not a perfect conductor but satisfies to first order the boundary condition $E_z=\pm\eta(\mu/\epsilon)^{1/2}$, where η is the reciprocal of the complex index of refraction of the plate relative to free space and is small for the range of conductivities considered. Through the introduction of a real current as source for the discontinuity of the tangential magnetic field and of a fictitious magnetic current as source for the discontinuity in the electric field, the author derives two Weiner-Hopf integral equations which may be solved independently. A careful study is made of the singularities of the various derivatives of the kernels, and of the correctness of the interchange of the Fourier transform and the limiting processes. Finally, the integrals for the resulting field are evaluated in the far zone by the method of stationary phase. Polarization of incidence both parallel and perpendicular to the edge is considered. With the exception of the derivation of the boundary condition to which reference is made elsewhere, the paper is completely self-contained.

W. K. Saunders (Washington, D. C.).

Bremmer, H. On the asymptotic evaluation of diffraction integrals with a special view to the theory of defocusing and optical contrast. Physica **18**, 469–485 (1952).

The problem of determining a function $u(x, y, z)$ which satisfies the partial differential equation $\Delta u+k^2u=0$ and equals a given function $u_0(x, y)$ in the plane $z=0$ has a unique solution in the half-space $z>0$ provided that the radiation condition at infinity is satisfied. The solution is, in fact,

$$u(x, y, z)=-\frac{1}{2\pi}\frac{\partial}{\partial z}\int\int u_0(\xi,\eta)\frac{e^{ikR}}{R}d\xi d\eta$$

where R is the distance from (x, y, z) to the integration point $(\xi, \eta, 0)$.

In practical problems, this double integral is extended only over a finite part of the plane $z=0$, since u_0 is always zero outside some closed boundary curve L. It is shown that this solution can be split up into two parts: (i) what may be called the geometrical-optical part which vanishes outside the cylinder through L with generators parallel to $0z$; and (ii) a diffraction part determined by the values of u_0 near L. Each part can be expanded into terms depending on $u_0(x, y)$ and on the functions

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^n u_0(x, y).$$

These expansions are asymptotic series for small values of k; the expansion of (i) is in powers of k^{-1}, of (ii) in odd powers of $k^{-1/2}$. The terms of the diffraction part consist of integrals along the curve L, whereas the terms in the expansion of the geometrical-optical part do not depend on integrals. The first few terms of the latter part are essential for defocusing effects and can be connected with the brightness-contrast in the plane $z=0$.

E. T. Copson (St. Andrews).

Markov, G. T. Excitation of a circular wave-guide. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **22**, 747–758 (1952). (Russian)

Con riferimento a coordinate cilindriche il potenziale vettore (elettrico o magnetico) viene espresso per mezzo della corrente impressa j nella forma

$$A_z=\int_v j_zG dv,\quad A_r=\int_v[j_r\cos(\varphi'-\varphi)-j_\varphi\sin(\varphi'-\varphi)]Gdv,$$

$$A_\varphi=\int_v[j_\varphi\cos(\varphi'-\varphi)+j_r\sin(\varphi'-\varphi)]Gdv$$

essendo

$$\int_v\cdots dv=\int_{-\infty}^{+\infty}dz'\int_0^\infty r'dr'\int_0^{2\pi}\cdots d\varphi'$$

e G la funzione di Green, definita da

$$G(z',z;r',r;\varphi',\varphi)=\begin{cases}\dfrac{1}{8\pi}\sum_{n=-\infty}^{+\infty}e^{-in(\varphi'-\varphi)}\int_{-\infty}^{+\infty}H_n^{(2)}(\kappa r')J_n(\kappa r) \\ \qquad\times e^{\pm(\kappa^2-k^2)^{\frac{1}{2}}(z'-z)}\dfrac{\kappa d\kappa}{(\kappa^2-k^2)^{1/2}} \quad (r<r') \\ \dfrac{1}{8\pi}\sum_{n=-\infty}^{+\infty}e^{-in(\varphi'-\varphi)}\int_{-\infty}^{+\infty}J_n(\kappa r')H_n^{(2)}(\kappa r) \\ \qquad\times e^{\pm(\kappa^2-k^2)^{\frac{1}{2}}(z'-z)}\dfrac{\kappa d\kappa}{(\kappa^2-k^2)^{1/2}} \quad (r>r'). \end{cases}$$

Le formule vengono applicate all'eccitazione interna o esterna della guida d'onda cilindrica.

G. Toraldo di Francia (Firenze).

Lošakov, L. N. On the theory of propagation of waves in an electron beam. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **22**, 193–202 (1952). (Russian)

This paper considers the conditions under which an electron beam propagated in a wave guide will be amplified. Let E_z be the z-component of the field and J the current density of electrons. The problem is to find a solution of

$$\Delta E_z + \omega^2\epsilon\mu E_z = i\omega\mu J + \frac{i}{\omega\epsilon}\frac{\partial^2 J}{\partial z^2}$$

such that E_z satisfies the appropriate conditions on the walls of the guide and such that E_z behaves like $e^{\gamma z}$. From the equation of motion of the electrons J is proportional to E_z. With the help of this fact, it is easy to find a fourth-order equation satisfied by γ. One root of this equation is close to the value of γ when an electromagnetic wave without an electron beam is propagated down the wave guide. After this root is removed from the equation, the resulting cubic is solved by Cardan's method and the conditions under which complex roots, and therefore amplification, is obtained are considered.

B. Friedman.

Ledinegg, E., und Urban, P. Zur Theorie der Hohlrohrwellen. Acta Physica Austriaca **5**, 1–11 (1951).

This is a generalization of an earlier paper by Ledinegg [Ann. Physik (5) **41**, 537–566 (1942); these Rev. **5**, 69] on the general solution of Maxwell's equations in a closed cylindrical space, such as a cavity resonator. In the present paper the authors discuss the completeness of the solution in normal modes for an infinitely long cylindrical guide by assuming that the guide may be subdivided into a sequence of resonators whose transverse boundaries are defined by the points on the z-axis at which the tangential derivative $\partial E_z/\partial z$ vanishes. For any finite section the former theory holds; for an infinite section at either end of the guide the addition of the physically obvious restriction that the longitudinal field components must be finite at $|z|=\infty$ is sufficient to insure that the field is completely defined by the normal mode solutions. (See also a recent paper by J. Van Bladel [J. Appl. Phys. **22**, 68–69 (1951); these Rev. **12**, 776].)

M. C. Gray (Murray Hill, N. J.).

Ledinegg, Ernst, und Urban, Paul. Über die Vollständigkeit der Hohlrohrwellen des *E*- und *H*-Typs. Arch. Elektr. Übertragung **6**, 109–113 (1952).

This is a somewhat shorter version of the paper reviewed above.

M. C. Gray (Murray Hill, N. J.).

Ledinegg, E., und Urban, P. Zur Theorie der erzwungenen Schwingungen elektrodynamischer Systeme. Acta Physica Austriaca **5**, 510–528 (1952).

The authors consider electromagnetic oscillations inside a perfectly conducting cavity resonator, energized over a small portion of the boundary surface by means of a coaxial line. Thus the boundary condition for the resonator is that $E_t=0$ over the surface except over the energized section. If the actual field over this section is known, the field inside the resonator can be expressed as sums of the normal mode solutions for free oscillations, multiplied by constant factors determined from the assigned field.

As a practical example two methods of energizing a circular cylindrical cavity are analyzed in detail. "Electric excitation" is obtained by using a linear antenna extending inside the resonator from the inner coaxial conductor. Then it is possible to measure the voltage and current at some point on the external part of the coaxial line and to express the expansion constants explicitly in terms of the field at the corresponding cross-section. Similarly "magnetic excitation" is obtained by introducing a small loop perpendicular to the direction of the magnetic lines of force between the inner and outer conductors, which induces a definite voltage at the ends of the coaxial line.

The authors also discuss in very general form the problem of reducing a differential equation, $\mathbf{D}(\mathfrak{E})=g$, where $\mathbf{D}$ is a self-adjoint vectorial differential operator and g a given vector field, to a vectorial integral equation, $\mathfrak{E}=\int(K_{r,s})gd\tau$, where $(K_{r,s})$ is the symmetric matrix of a "Green's tensor" K, and $\mathfrak{E}$ satisfies prescribed homogeneous boundary conditions over a closed surface. For the cavity resonator it is shown that the components of K can be expressed in terms of the normal mode solutions for free oscillations.

M. C. Gray (Murray Hill, N. J.).

Gercenštein, M. E. On longitudinal waves in an ionized medium (plasma). Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 303–309 (1952). (Russian)

Viene calcolata la costante dielettrica di un plasma elettronico, tenendo conto della dispersione termica della velocità degli elettroni. Se $f(\mathbf{v})$ è la funzione di distribuzione delle velocità nel plasma omogeneo e isotropo, la corrente di conduzione provocata da un'onda elettromagnetica di frequenza ω risulta

$$\mathbf{j} = -i\frac{e^2}{m}N\int\frac{(\mathbf{E}\times\mathbf{v})\times\mathbf{k}+\omega\mathbf{E}}{(\omega-\mathbf{k}\cdot\mathbf{v})^2}f(\mathbf{v})d\mathbf{v}$$

avendo indicato con $\mathbf{k}$ il vettore d'onda e con N la densità degli elettroni. Se ne ricava subito la costante dielettrica. Essa dipende dalla direzione di $\mathbf{k}$ e quindi il mezzo è anisotropo. L'autore esamina in particolare le onde longitudinali che possono sussistere nel mezzo.

G. Toraldo di Francia (Firenze).

Foldy, L. L. The electromagnetic properties of Dirac particles. Physical Rev. (2) **87**, 688–693 (1952).

A phenomenological framework for the description of the interaction of a Dirac particle with a weak, slowly varying external electromagnetic field is given. The interaction terms are given as an infinite series in the derivatives of the external potential evaluated at the particle position. A physical interpretation of the coefficients of the series in terms of moments is found by transforming to the non-relativistic Foldy-Wouthuysen representation. Limitations of the results and comparison with field theories are discussed.

K. M. Case (Ann Arbor, Mich.).

Kucera, Jaroslav. Les repères tournants dans l'analyse tensorielle des machines électriques. Rev. Gén. Électricité **61**, 325–338 (1952).

The dynamical equation of Lagrange as well as certain specialized concepts of holonomic and non-holonomic reference frames are reviewed as they apply to rotating electrical machinery. The theory of the "primitive" machine, from which all other machines may be derived by a group of transformation matrices, is discussed in greater detail. The salient pole synchronous machine is studied as a practical illustration. *G. Kron* (Schenectady, N. Y.).

Quantum Mechanics

Bastin, E. W., and Kilmister, C. W. The analysis of observations. Proc. Roy. Soc. London. Ser. A. **212**, 559–576 (1952).

The authors attempt to give an abstract axiomatic treatment of some of the problems treated by Eddington in his book "Fundamental theory" [Cambridge, 1946; these Rev. **11**, 144]. They introduce a set of axioms to cover observables and the process of measurement. These are quite different from those of quantum theory. The authors then define particles and discuss the hydrogen atom. This reviewer could not follow the latter discussion because there is no argument given for associating the mathematical statements made with the physics of the hydrogen atom, and because quantities such as μ, rest mass, are referred to in the discussion and never defined. *A. H. Taub* (Urbana, Ill.).

Géhéniau, J. Espace du noyau et structure en couches. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 71–77 (1952).

The author considers a Euclidean 4-space and in it the hypersphere $x^2+y^2+z^2+u^2=R^2$. He identifies the projection of this hypersphere on $u=0$ with the interior of an atomic nucleus, a Riemannian metric being induced in it from the hypersphere, so that it is a semihyperspherical 3-space V_3. Considering a nucleon as a free particle in V_3, he studies the eigenfunctions of the equation $(2m_0)^{-1}\Delta\psi+E\psi=0$, Δ being the Laplacian of V_3, and obtains by use of harmonic polynomials in x, y, z, u the eigenvalues $E_n=n(n+2)/2m_0R^2$. Azimuthal quantum numbers are also treated. *J. L. Synge* (Dublin).

Liesse, Cl. Sur la théorie des couches nucléaires. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **38**, 176–181 (1952).

Following a suggestion made in the paper reviewed above, the author takes, instead of a hypersphere, a hyperellipsoid $x^2+y^2+z^2+au^2=R^2$, and so treats the nucleus as a semihyperelliptical space. The method of harmonic polynomials is no longer available and polar coordinates are used. The case $a=1$ is first worked out and then an approximation for the case where $(a-1)/a$ is small; for this the energy levels are

$$E_{nl}=(2m_0R^2)^{-1}\{n(n+2)+[(a-1)/2a]\times[(n+1)^2-l(l+1)(n^2+2n+2)/n(n+2)]\}.$$

The case where a tends to infinity is also treated, giving

$$E_{nl}=(2m_0R^2)^{-1}\{\xi[\tfrac{1}{2}(n-l+1),l]\}^2$$

where $\xi[i,l]$ is the ith zero of the Bessel function of order $l+\frac{1}{2}$. In both papers connection is made with the scheme of M. G. Mayer [Physical Rev. **78**, 16–21, 22–23 (1950)]. *J. L. Synge* (Dublin).

Datta Majumdar, Sudhansu. The problem of three bodies in quantum mechanics. Z. Physik **131**, 528–537 (1952).

The reduction of the number of independent variables in Schrödinger's equation for interacting particles from $3N$ to $3N-6$ given by Wigner and Breit's method using the integrals of motion is worked out by the author in greater detail. This method can be used to calculate the wave functions and term values of helium with high precision. The discussions are strictly confined to a system of three particles but most of the results can be extended to a system consisting of any number of particles. The author gets extensions of the results of Breit [Physical Rev. **35**, 569–578 (1930)] in several important respects. First, it is shown by using Whittaker's coordinates that for given values of the quantum numbers l and m the eigenfunctions belong to two distinct classes completely independent of each other, one involving only odd values of the summation index τ in the expansion of the wave function of the problem and the other only even values. The expansion of the wave function contains the $2l+1$ eigenfunctions U_τ of the symmetrical top. Secondly, the effect of operation by the Hamiltonian on the top functions is studied in detail. The theory is applicable equally to the helium atom and the ionized hydrogen molecule H_2^+. The D-states of helium without nuclear motion are discussed in the last section of the paper. *M. Pinl* (Dacca).

Hanus, Wanda. The torsional oscillator. Acta Phys. Polonica **10**, 173–192 (1951).

A quantum mechanical theory of the torsional oscillator is developed for the purpose of extending the theory of Pauling [Physical Rev. (2) **36**, 430–443 (1930)] and Stern [Proc. Roy. Soc. London. Ser. A. **130**, 551–557 (1931)] to molecules possessing three different principal moments of inertia. The calculations have been carried out on the assumption that the amplitudes of these vibrations are small. *C. Kikuchi* (East Lansing, Mich.).

Nambu, Yoichiro. On Lagrangian and Hamiltonian formalism. Progress Theoret. Physics **7**, 131–170 (1952).

An extensive and formal investigation of the methods of deriving a Hamiltonian from a Lagrangian. Both classical and quantum dynamics are considered, many special one-coordinate systems being treated as examples. The methods are extended to Lagrangians containing higher time-derivatives than the first. The results are too miscellaneous to be summarized here. *F. J. Dyson* (Ithaca, N. Y.).

Bonč-Bruevič, V. L., and Medvedev, B. V. On the invariant construction of a quantum theory of fields. II. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 425–435 (1952). (Russian)

The authors start by writing down explicitly the most general possible formalism describing with Hamiltonian equations of motion a finite collection of quantized intersecting scalar fields. [See the earlier paper of the authors with N. N. Bogolyubov, Doklady Akad. Nauk SSSR **74**, 681–684 (1950); these Rev. **12**, 464.] They then derive the conditions to be satisfied by the arbitrary functions appearing in the formalism in order that the whole theory be (i) Lorentz-invariant and (ii) localizable. The latter requirement means that the Hamiltonian should be expressible as the integral of an energy-density operator, which is itself a function of the field quantities and their derivatives at a single point of space-time. The necessary conditions are formulated in a general and not impossibly cumbersome

form. However, the authors point out that in these conditions there appear integrals over momentum-space which are in general divergent. It is proved that such divergences actually appear in every Lorentz-invariant and localizable theory, except for the trivial case of non-interacting fields. Therefore, the authors say, there are no Lorentz-invariant and localizable theories of interacting fields; because in any theory which would apparently be Lorentz-invariant and localizable it is impossible even to formulate the condition for Lorentz-invariance unambiguously. The reviewer does not agree with this sweeping statement of the author's conclusions. A theory which is formally Lorentz-invariant and localizable, and which contains divergences, is still a theory. Nevertheless the actual result of this paper, that in every such theory some divergences necessarily exist, is valid and important. *F. J. Dyson* (Ithaca, N. Y.).

Dirac, P. A. M. A new classical theory of electrons. II. Proc. Roy. Soc. London. Ser. A. **212**, 330–339 (1952).

D. Gabor having pointed out that the author's recent theory [same Proc. **209**, 291–296 (1951); these Rev. **13**, 893] does not allow for some fields which apparently occur in nature, Dirac generalizes it by introducing two new variables ξ, η such that the electromagnetic potentials $A^\mu = kv^\mu + \xi\partial_\mu\eta$ where v^μ is a velocity flow and $k = m/e$ is a universal constant as before. The term $\xi\partial_\mu\eta$ did not occur previously. The revised theory is formulated in terms of two equivalent Action Principles and the Hamiltonian for the second is obtained. *A. J. Coleman* (Toronto, Ont.).

Cini, M. A perturbation method for Dirac's new electrodynamics. Proc. Roy. Soc. London. Ser. A. **213**, 520–529 (1952).

Since e does not appear explicitly in Dirac's new classical theory, the usual perturbation method of electron theory is not available. An alternative method of successive approximations is outlined based on the assumption that the fields arising from small charges may be neglected in comparison with the vacuum field. The method is not very practical if $\partial_\mu\xi\partial_\nu\eta - \partial_\nu\xi\partial_\mu\eta$ is large. *A. J. Coleman* (Toronto, Ont.).

Hoffmann, Banesh. Dirac's new classical theory of electrons. Physical Rev. (2) **87**, 703–705 (1952).

An analogy is shown to exist between the equations of the unrevised form of Dirac's recent theory [see second preceding review] and some of the equations of a projective theory of relativity formulated by the author [Physical Rev. **72**, 458–465 (1947); these Rev. **9**, 107]. He concludes ". . . gravitational effects . . . may not be negligible in a theory of electrons of the sort proposed by Dirac".
A. J. Coleman (Toronto, Ont.).

Novožilov, Yu. V. Application of Fok's method of functionals to the problem of self-energy. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **22**, 264–275 (1952). (Russian)

The content of this paper is a rather thorough discussion of the equations describing an electron in quantum electrodynamics, including the interaction of the electron with an external field and with its own self-field. The wave-function of the system is expressed as a function of electron and photon coordinates by the method of V. A. Fok [Phys. Z. Sowjetunion **6**, 425–469 (1934)]. The meaning and correct treatment of the negative-energy components of the electron wave-function are carefully explained. Finally, the equations are solved to second order in the radiation field interaction, and expressions obtained for the radiative correction to the energy of a bound electron, in agreement with previous authors [N.M. Kroll and W. E. Lamb, Physical Rev. (2) **75**, 388–398 (1949)]. It is shown how the infinite mass-renormalization effect can be unambiguously identified by expressing the operators in relativistically invariant form, so that the finite observable energy shifts become well-determined. *F. J. Dyson* (Ithaca, N. Y.).

Corben, H. C. A reformulation of field theory. Nuovo Cimento (9) **9**, 580–596 (1952).

The proposed reformulation of field theory is related to the five-dimensional unified field theory recently proposed by the author [Nuovo Cimento **9**, 235–252 (1952); these Rev. **13**, 712], but it is developed independently within the framework of the special theory of relativity. An auxiliary variable $x_5 = u$ is introduced, and the wave function of a particle of charge e is written in the form $\psi(x_\mu) = \psi(\mathbf{r}, t) \exp[iaeu]$ $(\mu = 1, \cdots, 5)$ where a is a constant. Both quantization and conservation of charge are deduced from the postulate that all field variables are periodic in u. Explicit use of the concept of gauge invariance of the first kind is then no longer necessary. When applied to a complex scalar field, the new theory appears to be equivalent with the usual formalism for particles of a given fixed charge, but for particles of spin $\frac{1}{2}$ it leads to a generalized Dirac equation which comprehends a mixture of charged and neutral fields. The mass of the charged particles is larger than the mass of the neutral particles in this theory which can thus be used to describe the electron and the neutrino as different states of one particle, but not the proton and neutron. However, one consequence of this theory is that there should exist multiply charged particles, for which there is no definite experimental evidence. *E. Gora* (Providence, R. I.).

Rayski, Jerzy. On the divergence problem in the theory of quantized fields. Acta Phys. Polonica **9**, 87–98 (1948).

The author proposes an "extended source" model of a charged particle, in order to avoid the divergence of the particle self-energy in quantum electrodynamics. He shows how such an extended source can be made relativistically invariant, at the cost of strict causality over short periods of time. His proposal, made first in 1946, has been independently put forward by Peierls and McManus [Proc. Roy. Soc. London. Ser. A. **195**, 323–336 (1948); these Rev. **10**, 664]. The physical interpretation of this model raises various difficulties which have not yet been satisfactorily overcome [see the following reviews]. *F. J. Dyson.*

Rayski, Jerzy. On the theory of non-local fields. Acta Phys. Polonica **10**, 103–105 (1950).

This is a comment on the non-local field theory of Yukawa [Physical Rev. **77**, 219–226 (1950); these Rev. **11**, 567]. The author points out that a Yukawa non-local scalar field is physically indistinguishable from an ordinary localizable field, so long as the field does not interact with anything. When interaction is present, the non-local field behaves like a localizable field except that the interaction is spread out over a finite volume instead of occurring at a point. The non-local theory with interaction is thus equivalent to the proposal for a theory of localizable fields with extended-source interactions [see the preceding review]. The particle self-energies will be finite just as in the author's theory.
F. J. Dyson (Ithaca, N. Y.).

Rayski, Jerzy. Non-local quantum electrodynamics. Acta Phys. Polonica 10, 300–302 (1951).

The author applies the argument of an earlier letter [see the preceding review] to the Yukawa non-local version of quantum electrodynamics. He shows that here, as in the case of the scalar field considered before, the non-local theory is equivalent to a theory of local fields with modified commutation relations. He proposes to quantize the theory by following the method of Yang and Feldman [Physical Rev. 79, 972–978 (1950); these Rev. 12, 569]. *F. J. Dyson.*

Rayski, Jerzy. On field theories with non-localized interaction. Acta Phys. Polonica 11, 25–35 (1951).

The author investigates the behavior of the divergences in quantum electrodynamics when point interactions are replaced by extended sources. He uses two different modifications of the interaction. The first replaces the electromagnetic potential A_μ by an "averaged potential"

$$\bar{A}_\mu(x)=\int F(x-x')A_\mu(x')dx',$$

where $F(x-x')$ is a relativistically invariant source-function. This model was considered before by the author [see the third preceding review]. The second similarly replaces the electron-positron field $\psi(x)$ by an averaged $\bar{\psi}(x)$. He shows that neither modification alone is sufficient to eliminate divergences from the theory. However, both together are sufficient. The theory with both modifications is no longer gauge-invariant in the ordinary sense, and the law of conservation of charge is not valid inside limited regions of space-time. Conservation of energy, momentum, and charge holds only asymptotically over long intervals of time.

F. J. Dyson (Ithaca, N. Y.).

Rayski, Jerzy. On simultaneous interaction of several fields and the self-energy problem. Acta Phys. Polonica 9, 129–140 (1948).

Païs [Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. 19, no. 1 (1947); these Rev. 8, 554] and Sakata and Hara [Progress Theoret. Physics 2, 30–31 (1947)] have introduced a method of making the electron self-energy finite in quantum electrodynamics, by allowing the electron to interact with a hypothetical neutral scalar field in addition to the Maxwell field. The author here shows that a similar method succeeds in making the photon self-energy finite and equal to zero, as it should be. He makes the photon interact with one additional spinor field and two scalar fields in addition to the electron-positron field. He expresses the hope that this compensation of divergences can be extended so far as to achieve the removal of all divergences from the theory. Feldman [Physical Rev. 76, 1369–1375 (1949); these Rev. 12, 150] has however subsequently shown that such a complete compensation is not possible.

F. J. Dyson (Ithaca, N. Y.).

Rayski, J., and Rzewuski, J. On a system of fields free of divergences of the mass-renormalization type. Acta Phys. Polonica 10, 159–172 (1951).

Rayski [see the preceding review] and independently Umesawa, Yukawa, and Yamada [Progress Theoret. Physics 3, 317–318 (1948)] have shown that the self-mass of photons vanishes if the electromagnetic field is coupled with spinor as well as with scalar charged fields. Now the question arises as to whether the divergences may be removed also from the remaining fields, i.e., whether the self-energy of the charged fields may be compensated by the same neutral C-meson field as was used for the electron self-energy, and whether both parts of the self-energy of the C-meson itself (due to couplings with electrons and with charged mesons) compensate to a finite value without introducing any further additional fields. The authors answer this question in the affirmative. They construct a simple system of five fields free of divergences of mass-renormalization type. The fields are: photon (mass 0), electron (mass m), neutral C-meson (mass $2m$), and two independent charged scalar meson fields (each of mass m). The couplings are: ordinary electromagnetic coupling (coupling constant e) between photon and the three charged fields, scalar coupling (coupling constant $2^{1/2}e$) between electron and C-meson, scalar coupling (coupling constant $2em$) between each charged meson and the C-meson. The complete compensation in this scheme is demonstrated only to second order in the coupling constants. In fourth order the complications would be vastly greater, and it seems unlikely that the compensation would still be complete. *F. J. Dyson* (Ithaca, N. Y.).

Rayski, Jerzy. Remarks on some non-linear effects in field theory. I. Acta Phys. Polonica 10, 151–158 (1951).

The author considers a system of two scalar fields interacting with a simple scalar interaction, one field having mass zero and the other a finite mass. He calculates by covariant perturbation theory the probability of mutual scattering of two mass-zero particles, occurring by virtue of the virtual production and annihilation of a pair of finite-mass particles. This is a simpler analog of the scattering of light by light in quantum electrodynamics. The result is finite. He also calculates the matrix element for the break-up of one mass-zero particle into two. *F. J. Dyson.*

Rayski, Jerzy, and Średniawa, Bronisław. Non linear effects in the theory of quantized fields. II. Acta Phys. Polonica 10, 207–212 (1951).

The authors calculated the scattering of light by light in the quantum electrodynamics of a charged scalar Klein-Gordon field. The method is the same as that used earlier by Rayski [see the preceding review]. The result quoted is not gauge-invariant, and is therefore in the reviewer's opinion incorrect. Details of the calculation are not given.

F. J. Dyson (Ithaca, N. Y.).

Rayski, Jerzy. A note on the invariant formulation of the quantum field theory. Acta Phys. Polonica 10, 29–31 (1950).

Tomonaga [Progress Theoret. Physics 1, 27–42 (1946); these Rev. 10, 226] set up a relativistically invariant Schrödinger equation by defining a wave-function with reference to an arbitrary space-like hyper-surface in Minkowski space-time, instead of making the wave-function a function of the time only. The author here observes that the Tomonaga formalism is more general than is necessary. To obtain a Lorentz-invariant Schrödinger equation it is sufficient to define the wave-function $\Psi(\Sigma)$ as a function of an arbitrary flat hypersurface Σ. Then the Schrödinger equation will take the invariant form

$$i(d\Psi(\Sigma)/d\Sigma)=\int_\Sigma d\sigma_\mu I_{\mu\nu}n_\nu\Psi(\Sigma).$$

Here $d\Sigma$ represents a small displacement of the flat hyper-surface parallel to itself, $d\sigma_\mu$ is a vector volume-element in the hyper-surface, and n_ν is the unit vector normal to the

hyper-surface. The tensor $I_{\mu\nu}$ is the energy-momentum tensor of the interaction, and is expressed in terms of interaction representation field operators. *F. J. Dyson.*

DeWitt, Bryce Seligman, and DeWitt, Cécile Morette. The quantum theory of interacting gravitational and spinor fields. Physical Rev. (2) 87, 116–122 (1952).

The Hamiltonian for the spinor field interacting with the gravitational field is derived. Firstly spinors in general coordinates are treated following the formalism of Pauli. The Hamiltonian is then constructed using the method of Pirani and Schild [same Rev. 79, 986–991 (1950); these Rev. 13, 306]. Since second class ϕ appear in the present case, a transformation of Poisson brackets is performed and after a suitable transformation of dynamical variables Dirac brackets are found in a form convenient for passage to quantum theory. The symmetrization of the Hamiltonian is then discussed. Finally a remark is made concerning the vacuum expectation value of the spinor stress density.
M. Suffczyński (Warsaw).

Eder, Gernot. Schwierigkeiten der Marchschen Theorie einer universellen Länge. Acta Physica Austriaca 5, 461–476 (1952).

Objections are raised to March's [e.g., same Acta 1, 19–41, 137–154 (1947); these Rev. 9, 320] use of a minimum length to remove divergence difficulties in the quantum theory of fields. Among points of criticism are: the conceptual basis of the geometry used, the definition of the density-function which replaces the Dirac delta-function, the ad-hoc nature of the interaction term in the Hamiltonian, and the discrepancy between calculated and experimental results. *C. Strachan* (Aberdeen).

Thermodynamics, Statistical Mechanics

Callen, Herbert B., and Greene, Richard F. On a theorem of irreversible thermodynamics. Physical Rev. (2) 86, 702–710 (1952).

This paper employs thermodynamic methods to re-establish a relation (connecting generalized impedance and fluctuations of generalized forces) previously obtained by statistical methods [Callen and Welton, same Rev. 83, 34–40 (1951); these Rev. 13, 477]. Various types of constraint are considered. *C. C. Torrance.*

Surinov, Yu. A. On the functional equations of heat radiation in the presence of an absorbing and dispersive medium. Doklady Akad. Nauk SSSR (N.S.) 84, 1159–1162 (1952). (Russian)

This note is devoted to the derivation of some basic integral equations of the macroscopic kinetics of heat radiation of grey bodies which are separated by an absorbing and dispersing medium containing heat sources. The derivations are quite elementary and are based on a theory of the classification of radiation types considered in earlier papers of the author [same Doklady 72, 469–472 (1950); Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1950, 1345–1376 (1950); these Rev. 12, 467]. *H. P. Thielman.*

Bogoluboff, N. Method of functional derivatives in the statistical mechanics. I. Equilibrium theory. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. 1946, no. 8, 177–190 (1947). (Ukrainian. Russian and English summaries)

Consider a system of N interacting particles in a volume V and let

$$U=\sum_{1\le i<j\le N}\Phi(|r_i-r_j|)$$

be the potential energy of the system (r_i is the position vector of the ith particle and $|r_i-r_j|$ is the distance). Let

$$Q_N=\int\cdots\int e^{-U/\theta}dr_1\cdots dr_N,\quad \theta=kT,$$

be the partition function. The distribution function of a number of specified particles is defined as the integral of $D_N=Q_N^{-1}e^{-U/\theta}$ over all variables except those associated with the specified particles. The purpose of this paper is to suggest a formal method of evaluating these distribution functions. Briefly, the method is as follows: Let $u(r)$ be an "arbitrary function" and consider the functional

$$L_N(u)=\int\cdots\int D_N\prod_{i=1}^{N}\left(1+\frac{V}{N}u(r_i)\right)dr_1\cdots dr_N.$$

The first variation of $L_N(u)$ can be written in the form

$$\delta L_N(u)=V\int\cdots\int D_N\prod_{2}^{N}\left(1+\frac{V}{N}u(r_i)\right)\delta u(r_1)dr_1\cdots dr_N.$$

The author now defines the formal "functional derivative" as

$$\frac{\delta L_N}{\delta u(r_1)}=V\int\cdots\int D_N\prod_{2}^{N}\left(1+\frac{V}{N}u(r_i)\right)dr_2\cdots dr_N$$

and notes that as $u\to 0$ one gets the distribution function of the first particle. Higher distribution functions are expressible in terms of higher functional derivatives. Various relations between functional derivatives are established and then applied to the case of short range forces. One is led then to the familiar Ursell-Mayer expression. *M. Kac.*

Bogolyubov, M. M. The equations of hydrodynamics in statistical mechanics. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. 1948, no. 10, 41–59 (1948). (Ukrainian. Russian summary)

An alternative derivation of equations of hydrodynamics based on the author's previous work. In the present paper only the first approximation (ideal fluid) is considered but apparently the method can be applied to higher approximations as well. The whole treatment is sufficiently complicated to make one wonder in what respect it is superior to the usual one based on the Boltzman equation.
M. Kac (Ithaca, N. Y.).

Klein, Martin J. The ergodic theorem in quantum statistical mechanics. Physical Rev. (2) 87, 111–115 (1952).

The author discusses the equality of time and ensemble averages (ergodicity) in quantum statistical mechanics. The problem is formalized by considering a system in interaction with its surroundings. Under suitable hypotheses on the interaction, conditions on the operators of the overall system in their action on the given system are found which are necessary and sufficient for ergodicity. These conditions are reminiscent of the condition of metric transitivity in the classical case. *J. L. Doob* (Urbana, Ill.).

Bergmann, Peter G. Generalized statistical mechanics. Physical Rev. (2) **84**, 1026–1033 (1951).

The author points out that earlier attempts to construct a relativistic thermodynamics [A. Einstein, Jbuch Radioaktivität und Elektronik **4**, 411–462 (1908); M. Planck, S.-B. Preuss. Akad. Wiss. **1907**, 542–570; Ann. Physik (4) **26**, 1–34 (1908); R. C. Tolman, Relativity, thermodynamics, and cosmology, Oxford, 1934; C. Eckart, Physical Rev. (2) **58**, 919–924 (1940)] have been of a phenomenological character. He believes that one must begin with a formulation of a relativistically invariant statistical mechanics, independent of the idea of thermal equilibrium. A satisfactory form of relativistic thermodynamics will then follow.

The author begins by making the time one of the canonical variables. As a consequence, "stationary ensembles no longer occupy a privileged position among all conceivable Gibbs ensembles, and it becomes necessary to redefine and reformulate most statistical and thermodynamic concepts." After laying down a set of covariant definitions of canonical ensembles, entropy, temperature, heat flux, and performance of work, he proves a theorem analogous to the usual H-theorem, and a generalization of the Second Law of Thermodynamics. *A. W. Sáenz* (Bloomington, Ind.).

Broer, L. J. F. On the dynamical behaviour of a canonical ensemble. Physica **17**, 531–542 (1951).

The author investigates the effect caused by an adiabatic variation of an external parameter on a canonical ensemble. When the classical equations of motion are used, the adiabatic change of the system is correctly obtained. But using quantum mechanics he encounters difficulties which are not satisfactorily explained. [The matter has been clarified recently by M. J. Klein, Physical Rev. **86**, 807 (1952).] *F. London* (Durham, N. C.).

Verschaffelt, J. E. Théorie des phénomènes de transport basée sur le principe de superposition. Physica **18**, 43–62 (1952).

The author organizes and summarizes the work contained in the bibliography of thirty-two of his papers published during the last seventeen years. Considering a fluid mixture, he calls gradients of generalized potentials (temperature T, pressure, concentration, gravitation potential, electric potential, magnetic potential) simple affinities $\mathbf{A}_\lambda$. The rate of dissipation of energy is

$$\epsilon=-\mathbf{w}\cdot\operatorname{grad}\log T+\sum \mathbf{m}_\nu\cdot\mathbf{A}_\nu$$

where $\mathbf{w}$ is the thermal flux, $\mathbf{m}_\nu$ is the flow vector for the νth component, and $\mathbf{A}_\nu$ is the "complex affinity." He supposes both $\mathbf{A}_\nu=\sum s_{\nu\lambda}\mathbf{A}_\lambda$ and

$$(*)\qquad \mathbf{w}=k\operatorname{grad}\log T,\quad \mathbf{m}_\nu=a_\nu\mathbf{A}_\nu,$$

thus obtaining for ϵ a quadratic form in the simple affinities. The assumption (*) he calls his "principle of superposition," which he regards as replacing Onsager's reciprocity relations. The author indicates the application of these ideas, fortified by further hypotheses, in various parts of mechanics, physics, and chemistry. *C. Truesdell.*

Nijboer, B. R. A., et van Hove, L. Sur la fonction de distribution radiale d'un gaz imparfait et le principe de superposition. Nederl. Akad. Wetensch. Proc. Ser. B. **54**, 256–259 (1951).

The radial distribution function $g(r)$ can be expanded in powers of the density ρ:

$$g(r)=\exp[-V(r)/kT][1+\rho g_1(r)+\rho^2 g_2(r)+\cdots];$$

$V(r)$ is the intermolecular potential. For elastic spheres, Kirkwood computed

$$g_1(r)=\begin{cases}\frac{2}{3}\pi(2-\frac{3}{2}r+\frac{1}{8}r^3) & \text{for } r\leqq 2,\\ 0 & \text{for } r\leqq 2.\end{cases}$$

For the same case, the authors state

$$g_2(r)=\tfrac{1}{2}[g_1(r)]^2+\varphi(r)+2\psi(r)+\tfrac{1}{3}\chi(r),$$

where φ, ψ, and χ are lengthy but explicit expressions in terms of elementary functions; in particular $g_2=0$ for $r\geqq 3$. If the simplifying "superposition" assumption of Kirkwood is made, it is found that g_1 is unchanged but g_2 is considerably in error; consequently, doubt is cast upon the quantitative accuracy of this assumption for large density, in particular, near condensation. *H. Grad.*

Nijboer, B. R. A., and Van Hove, L. Radial distribution function of a gas of hard spheres and the superposition approximation. Physical Rev. (2) **85**, 777–783 (1952).

Essentially the same as the above paper, but containing derivations. *H. Grad* (New York, N. Y.).

Dutta, M. On equation of state of real gases. Proc. Nat. Inst. Sci. India **18**, 81–91 (1952).

The author obtains a modified form of the Planck-Saha-Bose equation of state by an improved treatment of the overlapping of volumes of exclusion. In this form the second virial coefficient is 30% larger than its usual value. *C. C. Torrance* (Monterey, Calif.).

Truesdell, C. On the viscosity of fluids according to the kinetic theory. Z. Physik **131**, 273–289 (1952).

From the author's introduction: "In this note I wish to discuss two of the celebrated results of Maxwell for gases: (1) the two viscosity coefficients λ, μ are connected by the "Stokes relation" $3\lambda+2\mu=0$, (2) the viscosity μ is independent of density but is proportional to the square root of the temperature. In the first part I draw attention in particular to the predominant effect of the choice of the basic definitions of stress and temperature—an effect so strong that in a sense one may say that the kinetic theory assumes rather than proves that $3\lambda+2\mu=0$ or $3\lambda+2\mu\neq 0$, as the case may be. In the second part I show from dimensional considerations that the very idea of a molecular model for a fluid carries with it the assumption that both the viscosity coefficients depend but weakly on the density, and that only by supposing the inter-molecular forces very strong can we obtain any other dependence upon temperature than an approximate proportionality to its square root."

In the opinion of the reviewer, insufficient appreciation is shown in the first part of the paper of the inherent limitations of thermodynamical concepts in non-equilibrium. The reviewer is unable to follow the arguments of the second part in entirety. Since the viscosity coefficients are not in general proportional to the square root of the temperature, it is not clear what is the significance of the above quotation concerning this point. *H. Grad* (New York, N. Y.).

Kihara, Taro. The second virial coefficient of non-spherical molecules. J. Phys. Soc. Japan **6**, 289–296 (1951).

The Lennard-Jones molecular model implies spherical symmetry and assumes an intermolecular potential of the form

$$(*)\qquad U(\rho)=U_0\left[\frac{m}{n-m}\left(\frac{\rho_0}{\rho}\right)^n-\frac{n}{n-m}\left(\frac{\rho_0}{\rho}\right)^m\right]$$

containing as adjustable parameters a length ρ_0, a potential U_0, and two exponents n and m. The author considers four, more general, nonspherical models with an additional length parameter l. First, assuming that the 'core' of a molecule is a line of length l, the potential between two like molecules in a given configuration is given by (*) where ρ is the shortest distance between cores. Next the core is taken to be a disc of diameter l with the same potential (*), ρ again being the shortest distance between cores. The third and fourth models are ellipsoids (prolate or oblate) confocal with the core (line or disc), ρ being the length of minor axis for which the two molecules touch. As limiting cases ($n\to\infty$, $U_0\to 0$) these models include rigid 'spherocylinders' and ellipsoids. The author states (this is not obvious to the reviewer) that the second virial coefficient is given by

$$B(T)=\int_0^\infty [1-\exp(-U(\rho)/kT)]db_l(\rho),$$

where $b_l(\rho)$ is the virial coefficient for the corresponding rigid molecule of dimensions l and ρ (integration is with respect to ρ). The $b_l(\rho)$ are known explicitly [A. Isihara, J. Chem. Phys. **18**, 1446–1449 (1950)], and $B(T)$ is computed for the potential (*) by series expansion. Numerical results are given for $m=6$, $n=12$ and compared with experiment for H_2, N_2, C_2H_4, and CO_2. *H. Grad.*

Berlin, T. H., and Kac, M. The spherical model of a ferromagnet. Physical Rev. (2) **86**, 821–835 (1952).

This paper discusses the partition function for rectangular models analogous to the Ising model but with the spins σ_i located at the lattice points having different probability distributions. In the Gaussian model, the spins are independently and normally distributed with expected value zero and variance one. The spherical model is one in which spins such that $\sum\sigma_i^2=N$ are permitted and the a priori probability distribution is uniform over the N-dimensional sphere determined by this relation. In both cases explicit expressions for the partition function or its limit as N approaches infinity are obtained. The Gaussian model has an irrelevant singularity at $T\to 0$ but in the one- and two-dimensional case, the analytic form is analogous in certain ways with the known form of the Ising model. The spherical model does not have the low temperature difficulty. In the spherical models, there is no transition point for the one or two-dimensional model. In the three-dimensional model, the internal energy and specific heat are continuous but the temperature coefficient of the latter has a jump. This last is shown to be associated with a spontaneous magnetization. The magnetization is also discussed in the case of an applied magnetic field and for high temperatures formulas analogous to the Curie-Weiss formulas are obtained. In the appendices the mathematical discussion of the spherical model is given. This involves the spectral problem of the quadratic form which expresses the energy, a limiting process on an expression involving the characteristic roots of this form and the evaluation of a multiple definite integral by a saddle point method.

F. J. Murray (New York, N. Y.).

***Born, Max. Die Gültigkeitsgrenze der Theorie der idealen Kristalle und ihre Überwindung.** Festschrift zur Feier des zweihundertjährigen Bestehens der Akademie der Wissenschaften in Göttingen. I. Math.-Phys. Kl., pp. 1–16. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1951.

The usual quantum mechanical description of a crystal is based on certain assumptions concerning the position of the atoms on the lattice point and the author states that when the thermal expansion of a crystal is of the order of magnitude of the distance between atoms, then these assumptions are no longer valid. This indicates an upper bound for the linear dimensions of a crystal and because of the zero-point energy this upper bound is bounded for all temperatures by about 1000 atom distances. The author describes another approach to the quantum mechanical description of a crystal in which the equilibrium positions of the atoms in the crystal is not immediately specified and hence the result is free of the above difficulty. The process begins with the elimination of the electron motion from the Hamiltonian of the system yielding a Hamiltonian on the coordinates of the position of the atoms in which the equilibrium position coordinates appear as parameters. Instead of the differences $X-X_0$ between the coordinates of the position of the kernels and their equilibrium values X_0, new coordinates q are introduced which are the position variables for harmonic oscillators corresponding to the usual crystal oscillations. The condition that the mean position of q be zero determines X_0. The q's are linear in the $(X-X_0)$'s and their character determines the form of the Hamiltonian up to third powers of the $X-X_0$. The remainder of the potential function, i.e., the part dependent as third and higher powers of the q or $X-X_0$, is treated as a perturbation on the specified part of the Hamiltonian. The treatment is based on the Dirac density function for which a form has been given by Husimi for a system of oscillators. Thus Born obtains a density function ρ in the form $\rho_0 g$ where ρ_0 is that of Husimi and g consists of a sum of terms corresponding to the higher terms of the original potential function. In particular he shows the possibility of obtaining a lattice solution for the X_0's, but a constant appears which is undetermined, and thus the solution is not mathematically determined.

F. J. Murray (New York, N. Y.).

Nabarro, F. R. N., and Varley, J. H. O. The stability of hexagonal lattices with a simple law of force. Proc. Cambridge Philos. Soc. **48**, 316–328 (1952).

The binding energy per atom of a hexagonal lattice is considered to be in the form $\epsilon=3\varphi(a)+3\varphi(b)+\chi(V)$ where a and b are two edges of the fundamental cell and $V=\frac{1}{2}a^2(3b^2-a^2)^{\frac{1}{2}}$ is the volume. At equilibrium, ϵ regarded as a function of a and b is stationary. In general, there is such an equilibrium point a_0, b_0 with $a_0=b_0$, corresponding to a configuration of closely packed spheres. However "there may be another hexagonal structure of lower energy. A numerical example is given in which this occurs, binding energy and elastic constants being comparable with Zn ..." The example is based on an assumption concerning the form of $\chi(V)$, the portion of the energy dependent on the volume; the argument, involving various inequalities on the derivatives of φ and χ, is motivated by a graphical construction.

F. J. Murray (New York, N. Y.).

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